213 | 1.00

284 | 15.00

AN

ELEMENTARY TREATISE

ON

MECHANICS,

COMPREHENDING

THE DOCTRINE OF EQUILIBRIUM AND MOTION,

AS APPLIED TO

SOLIDS AND FLUIDS,

CHIEFLY COMPILED, AND DESIGNED

FOR THE USE OF THE STUDENTS OF THE UNIVERSITY

AT

CAMBRIDGE, NEW ENGLAND.

By JOHN FARRAR, LL. D.,

HOLLIS PROFESSOR OF MATHEMATICS AND NATURAL PHILOSOPHY.

SECOND EDITION, REVISED AND CORRECTED.

BOSTON:

HILLIARD, GRAY, AND COMPANY.

1834.

CAMBRIDGE:

CHARLES FOLSOM, PRINTER TO THE UNIVERSITY.

ADVERTISEMENT.

In selecting materials for this treatise particular regard has been had to the practical uses of the science; at the same time the theoretical principles are rigorously demonstrated. Where the nature of the subject admitted of it, the geometrical method has been preferred, as being more perspicuous, and better adapted to most learners. There are many refinements in the later and more improved treatises not adopted in this, for the reasons above mentioned, and on account of the insufficient provision, as to time and preparatory studies, that is made in most of the seminaries of the United States for a text-book upon such a plan.

The works principally used in preparing this treatise are those of Biot, Bézout, Poisson, Francœur, Gregory, Whewell, and Leslie. In the portions selected it was found necessary to make considerable alterations and additions in order to give a uniform character to the whole. There has often been occasion, moreover, in appropriating the substance of a proposition or course of reasoning, to amplify or condense it, or to vary the phraseology. It became inconvenient, therefore, to distinguish by quotations the respective portions taken from different authors. Bézout has been adopted, in substance, as the basis in what relates to statics, dynamics, and hydrostatics, although the matter is arranged

according to a different system; and Gregory, with many changes and substitutions, has been principally used in that which is comprehended under hydrodynamics.

This volume constitutes the first part of a course of Natural Philosophy. The second comprehends Electricity, Magnetism, and Electro-magnetism; the third Optics, and the fourth Astronomy. These subjects are treated in three different volumes.

Cambridge, Massachusetts,
September 30th, 1834.

CONTENTS.

Page.

Preliminary Remarks and Definitions . . . 1

STATICS.

Uniform Motion 13
Forces and the Quantity of Motion . . . 16
Equilibrium between Forces directly opposite . . 18
Compound Motion 20
Composition and Decomposition of Forces . . . 24
Moments and their Use in the Composition and Decomposition of Forces. 31
Parallel Forces which act in different Planes . . 38
Forces, the Directions of which are neither in the same Plane nor parallel to each other . . . 41
Centre of Gravity 43
Application of the Principles of the Centre of Gravity to particular Problems 50
Properties of the Centre of Gravity . . . 69
General Principle of the Equilibrium of Bodies . 75
D'Alembert's Principle, and Concluding Deductions . 76
Application of the Principles of Equilibrium to the Machines usually denominated Mechanical Powers . 79
Rope Machine 79
Lever 90
Pulley 102
Wheel and Axle 108
Inclined Plane 116
Screw 124
Wedge 128
General Law of Equilibrium in Machines . . 130

Friction 135
Stiffness of Cords 155
Balance, Steelyard, &c. 160

DYNAMICS.

Motion uniformly accelerated 165
Free Motion in heavy Bodies 168
Direct Collision of Bodies 175
Direct Collision of unelastic Bodies . . . 176
Force of Inertia 178
Application of the Principles of Collision of unelastic Bodies 181
Collision of elastic Bodies 185
Motion of Projectiles 190
Motion of heavy Bodies down inclined Planes . . 205
Motion along curved Surfaces 210
Motion of Oscillation 215
Pendulums 221
Line of swiftest Descent 225
Moment of Inertia 230
Centre of Percussion and Centre of Oscillation . 236
Actual Length of the Seconds Pendulum . . . 246
Measure of the Force of Gravity 249
Application of the Pendulum to Time-Keepers . . 250
Rotation of Bodies unconfined 257
Method of estimating the Forces applied to Machines . 265
Maximum Effect of Agents and Machines . . 273

HYDROSTATICS.

INTRODUCTORY REMARKS. 289

Pressure of Fluids 290
Solids immersed in Fluids 306
Specific Gravities 308
Spirit Level 321
Equilibrium of Floating Bodies 322
Capillary Attraction 336
Apparent Attraction and Repulsion observed in Bodies floating near each other on the Surface of Fluids . . 349
Barometer 351

HYDRODYNAMICS.

Discharge of Fluids through Apertures in the Bottom and Sides of Vessels 369
Motion of Gases 377
Resistance of Fluids to Bodies moving in them . . 388
Theory of the Air-Pump 399
Pumps for Raising Water 402
Syphon 412
Steam-Engine 413

NOTES 427

AN

ELEMENTARY TREATISE

ON

MECHANICS.

Preliminary Remarks and Definitions.

1. MATTER has been variously defined by philosophers, and some have even doubted whether we can be morally certain of its existence. It is not our intention, nor does it belong to the nature of our subject, to enter into discussions of this kind. Relying solely upon experiment, we give the name of *matter* or *body* to whatever is capable of producing, through our organs, certain determinate sensations; and the power of exciting in us these sensations constitutes for us so many *properties*, by which we recognise the presence of bodies. But among these properties, two only are absolutely essential in order to our having a perception of matter. These are *extension* and *impenetrability*, of which the sight and touch are the first judges.

2. The character derived from extension is self-evident; when we see or touch a body, this body, or, if you please, the power which it has of affecting us, resides in a certain portion of space. The place which it occupies is therefore determinate; and by this very circumstance it is extended.

3. When we pass our hands over the surface of a body, we perceive that the matter of which it is composed, is without us; moreover, two distinct portions of matter can never be made to coincide or identify themselves, the one with the other, in such a manner that the same absolute points of space shall at the same time give us the sensation of both. In this consists the property of impenetrability.

To show how this property, together with that of extension, is necessary to constitute a body, I will refer to familiar phenomena in which these properties are observed separately.

If an object be placed before a concave mirror, there will be formed, at a certain distance from the mirror, an image of the object. This image, distinct from the parts of space that surround it, is extended but not impenetrable. The hand may be thrust through it without experiencing the smallest resistance, and the parts that come in contact with the hand, vanish instead of being displaced. A piece of wood or stone does not admit of being thus penetrated. Moreover, by means of a second mirror properly disposed, an image of another object may be made to occupy the same place with that of the first, without the latter being displaced, or in any way deranged. Indeed the same coincidence may be effected with a third, a fourth, or any number of images. These images are extended, but not impenetrable; they are *forms*, but not *sensible matter.* I say *sensible* matter, for we shall see hereafter that light which constitutes these images, is itself probably composed of material particles of an insensible tenuity, which move with amazing velocity, and only pass by each other in this case at immense intervals, by which they are separated from each other.

4. It is here proper to speak of certain phenomena which seem, at first sight, to be opposed to what we have laid down with regard to the impenetrability of matter, but which, examined more attentively, only tend to confirm it.

When a solid body is suffered to fall into any fluid, as water for example, it sinks and seems to penetrate the fluid; but it in fact only separates and displaces the parts that compose it; for if the vessel containing the fluid be formed with a narrow neck toward the top, like a bottle, the fluid will be seen to rise as the body enters, and to a greater or less height, exactly in proportion to the size of the immersed body. What has taken place therefore, is only a division and separation of parts, and not strictly a penetration. The same may be said when an edged tool is forced into a block of wood, only the parts of the wood are separated with more difficulty than those of water. The same may be said also when a nail is driven into clay, lead, or gold, in which cases it only makes an opening sufficient for its admission. Indeed the mass thus pierced is not entirely separated, but the parts are nevertheless pressed and

crowded together; and if we examine those which surround the opening, caused by the nail, we shall find sensible marks of this pressure. The nail in its turn may likewise be pierced by steel, and this again by other bodies.

We hence infer that bodies, even the most hard and compact, are not composed of matter absolutely continuous, but of parts aggregated together, and placed at distances, which, under the influence of external causes, may become greater or less. It is on this account that the dimensions of any given mass of matter are capable of being increased by heat, or diminished by cold, that the particles of salt admit of being separated and distributed, and as it were, lost among the particles of water; that mercury attaches itself to a piece of gold immersed in it, and insinuates itself into the interior of this compact substance. These mixtures and dissolutions sometimes take place without any apparent augmentation of bulk, this bulk being estimated according to the exterior surface of the bodies in question, without regard being had to the void spaces, sensible or insensible to us, which may be found to exist among their parts. In all this there is only separation and mixture without any actual penetration of material particles.

This want of material continuity in bodies is known under the general name of *porosity*, and we call *pores* the interstices or empty spaces by which these particles are separated from each other. Porosity seems to be a property common to all bodies, although it does not belong to the essence of matter, since we can conceive of sensible bodies which are entirely destitute of void space.

5. Thus admitting that bodies may be considered as composed of smaller parts which constitute their essence, we may be asked, what is the form and magnitude of these parts. As to the magnitude, it should seem that it is extremely minute; for to whatever extent we carry the division, in the case of gold, for example, by the processes of wire-drawing, filing, and beating, the smallest particles preserve invariably all the properties that belong to the entire mass. Crystallized bodies reduced to an almost impalpable powder, upon being examined with a microscope, are found to exhibit the same forms and the same angles which characterize the whole mass of the crystal. We have examples of a division carried to a still greater extent in odors, the sense being affected in this case by particles proceeding from the odoriferous body that are absolutely in-

visible and impalpable. From these few instances, and a thousand others that might be mentioned, it is evident that a body without changing its character, without ceasing to be of the same identical nature with the largest masses that surround us, may be divided into parts, the smallness of which eludes the power of the senses, and almost that of the imagination.

6. The question has been much discussed, whether matter be infinitely divisible; but it is now pretty generally agreed that the dispute is about words. If the point in question relate to abstract geometrical divisibility, there can be no doubt of the truth of the affirmative; for however infinitely small we suppose a particle, from the very circumstance of its being extended, we can always conceive this extent divided into two halves, and each of these into two others, and so on without end (*Calc.* 4). But if we mean by the question an actual physical divisibility, nothing can be decided absolutely one way or the other. It seems however by all we can learn, that we should at some stage of the division arrive at material particles which would not admit of being broken, or altered, or transmuted the one into the other; for to whatever chemical operation they are subjected, into whatever combinations they are made to enter, however they may be brought to constitute a part of living beings, they always return to their former state, with their original properties unchanged. The infinite variety of processes of this kind through which the same material particles have been made to pass since the world was created, does not appear to have produced the smallest alteration.

7. But how can such a system of particles exist collected together in the form of solid and resisting masses, as we see they are in a great number of bodies, in all indeed when they are properly examined? This state, as we shall see hereafter, is produced and maintained by the natural powers with which all parts of matter are endued, and which cause them to tend toward each other, *as it were by an attraction*. But if there existed only forces of this kind, the particles would continue to approach till they came into actual contact with each other, that is, until they were arrested by the impenetrability of their parts, which would not admit of the contraction and dilation which are constantly observed in bodies. We accordingly infer that there is a general cause of interior repulsion in bodies, by which the attractive forces are continually balanced. This

cause, which resides in all bodies, seems to be referable to the principle of heat. The particles of each body, actuated at the same time by these two opposite forces, naturally put themselves in a state of equilibrium, resulting from a compensation of energies, and they approach and recede, according as the forces to which they are exposed from without, favor the attractive or repulsive principle. It is with these minute bodies as it is with the planets of our system, which are found to move and oscillate, as it were, in orbits of variable forms and dimensions, without the system being destroyed, or the general equilibrium being disturbed. From these different conditions of equilibrium arise, as we shall see hereafter, all the secondary and changeable forms of bodies, such, for example, as are denominated aëriform, liquid, solid, crystallized, hard, elastic, &c.

8. In all the phenomena which present themselves, the particles of matter act, or rather are acted upon, as if they were perfectly *inert*, that is, deprived of all power of self-direction. They can be moved, displaced, stopped, by causes foreign to themselves; but we never have been able to discover the least trace of any thing like choice or will, proper to the particles themselves. If the ball which rolls upon a billiard-table, in consequence of the impulse that is given to it, loses by little and little its velocity, and at length comes to a state of rest, it is entirely the effect of the continual resistance that it meets with from the roughness of the cloth with which it comes in contact, and from the particles of the air through which it passes. Make the cloth more smooth, or the air more rare, and the same impulse would keep it longer in motion; substitute for the cloth a marble slab highly polished, or a band of stretched wire, the elasticity of which is still more perfect, and the ball would continue its motion for a much longer time; from all which it is to be inferred, that if the obstacles were completely removed there would be no diminution of the velocity first communicated, and the motion would never cease. A stone thrown from the top of a tower, and urged at the same time by the impulse of the hand and by gravity, will come to the ground after proceeding a certain distance, losing at the same time its horizontal velocity, by imparting it to the particles of air against which it impinges. But let us suppose the air removed or annihilated, and the projectile force (in the direction of a tangent), to be sufficient to carry the stone as far from the earth as gravity would cause it to descend each instant, and the stone would describe a circle

round the earth, and if there were nothing to stop or obstruct it, it would thus continue to revolve without end. We have indeed this principle exemplified in the motion of the moon, which revolves in a void about the earth; we see moreover the same renewed, perpetual motion in the planets, which pass in like manner through spaces destitute of all material resistance. We are hence lead to believe that matter is incapable of effecting any change in itself, either with respect to motion or rest; and, once put into either of these states, it would continue in this state so long as it should remain undisturbed by any cause foreign to itself. This indifference to motion and rest, this want of all power of self-direction, has obtained the name of *inertia*. There is one class of bodies, however, that seems to form an exception to this law of matter. It comprehends those which we call *animated*, which put themselves in motion or stop themselves by an act of the will; but even in these the material elements which constitute their parts or members, and these members themselves, are perfectly inert. It is their union or combination that possesses the quality of life. Separated, they have no longer this power, but return to the condition of ordinary matter. We are entirely in the dark with regard to the cause of this remarkable difference in the bodies that surround us. As to what constitutes a state of life, we can pretend to no knowledge whatever. But seeing matter under all other circumstances destitute of the power of self-direction, and knowing also that in living beings it loses this faculty by death and by sleep, we are led to regard it as foreign to the essence of matter, and to consider the volition of animated beings, as the act of an immaterial principle which resides within them. We are unable to say in what part this principle is seated, or in what it consists, and still less how, being immaterial, it is capable of acting upon matter; but with the little attention that we have paid to ourselves and to the objects about us, these obscurities, unfortunately too common, in which our imperfect knowledge has left us, ought not to be made the grounds of an objection against the essence of things, with which we must be contented to remain unacquainted. So that we here proceed philosophically, according to the rule adopted in other cases, by bringing together things that are analogous, and making the motion of animated beings to depend upon a cause foreign to matter; matter being found inert under all other circumstances in which we have been able to examine it. Another reason is given in the schools of philosophy for attributing spontaneous

motion to an immaterial principle; namely, that the will, by the very nature of its acts, can proceed only from a simple being, and that consequently it cannot belong to a substance essentially compounded, or at least divisible and decomposable, like matter; but this metaphysical argument would carry us too far from our subject. We content ourselves with merely suggesting it; for all experimental purposes, it will be sufficient to consider the immateriality of the principle of volition as a distinction founded upon analogy, and the *inertia* of matter as a general property in the actual state of the world.

9. We are moreover made acquainted by experiment, with several other properties of matter which are also accidental, that is, which seem not to be absolutely necessary in order that material bodies may manifest themselves to our senses, but the co-existence of which with the primitive conditions of materiality is important to be known, since it supplies the want of other evidence, in a great number of cases in which the essential properties do not admit of being recognised. Such, for example, is *gravity*. Among natural bodies which we can see and touch, none is to be found which is not heavy, that is, which does not tend to fall toward the centre of the earth when left to itself; and since these properties are always found to accompany each other, the presence of the one is with respect to us, always a sufficient ground to infer the existence of the other. Thus, although we can neither see nor touch the air, as we can see and touch other bodies, still we believe it to be a material substance, because it is heavy, capable of being confined in vessels and of exhibiting other phenomena, all similar to those which belong to a heavy fluid. A careful examination of these properties teaches us at length that there are airs of very different kinds, which are all so many substances differing essentially from each other in the action which they are capable of exerting on other bodies, and which is exerted in turn upon them by these bodies.

10. Moreover *attraction* is one of those contingent properties which supply what is wanting in the evidence furnished by the immediate testimony of the senses. I have said that the particles of all known bodies exert upon one another attractive and repulsive forces On the other hand, when we can demonstrate the existence of these forces in an unknown principle, we infer that this principle is material. Thus, *light* is not tangible; it is not, so far as we can

perceive, extended; it has no weight, or at least none capable of being appreciated by our balances. It is so subtile as to elude all the ordinary methods by which matter manifests itself to the senses. But by causing it to pass through transparent bodies, as glass, water, &c., it deviates from a direct course in its passage, and is bent precisely as if it were repelled by a force proceeding from the surface, and attracted on the other hand within by the particles which compose the transparent body. We know also that it employs a certain time, very short indeed, but yet capable of being estimated, in passing from luminous bodies to us. In fine, by subjecting rays of light to certain tests, we find that transparent bodies attract and repel them differently on certain sides from what they do on others. From these properties, taken together, we are led to conclude that light is a material substance, composed of particles extremely small, the form of which is symmetrical on certain faces, which are susceptible of particular attractions and repulsions, and which move in free space, and through transparent bodies, with a given and determinable velocity.

11. There are still other principles which act upon material bodies without being either visible or tangible, or susceptible of being weighed by our balances, which even present much fewer indications of materiality than light, and which notwithstanding are believed to be material substances. Such is the unknown principle of *electricity*. Nothing absolutely material has yet been detected in the cause of electrical phenomena, nothing indeed which does not admit of being explained without the supposition of matter. Still in its distribution over bodies, in its passage from one to the other through the obstacles which separate them, this principle acts in a manner so exactly conformable to the laws of equilibrium and motion which belong to fluid substances, that we can on this hypothesis calculate with the utmost precision, and in all their details, the phenomena that are to take place under given circumstances. It seems extremely probable, therefore, that the principle in question is a fluid, and that it is accordingly material. The same reasoning is applicable to the principle of *magnetism*, which manifests itself in several metals.

12. We have still less evidence of any thing material in the principle of *heat*. Not only does it want, like the preceding, the sensible properties by which matter is characterized, but the laws of

its motion and equilibrium not being completely known, we cannot arrive at the same probable conclusion in this case as in the former. By following it however in our experiments, we find that it diffuses itself in bodies, passes from one to another, modifies the disposition, the distances, and attractive properties of their particles. But all this does not prove incontestably, that the principle in question is itself a body. The strongest argument in favor of its materiality is derived perhaps from certain analogies, lately discovered, between the radiant properties of heat and those of light, which lead us to believe that one of these principles may change itself gradually into the other, that is, they may acquire and lose successively the modifications by which they are respectively distinguished. The developement of these analogies furnishes a most important subject of investigation.

13. It will be perceived from what has been said, that all bodies of a sensible magnitude, the materiality of which can be immediately determined, consist in the grouping together of a multitude of material particles of extreme minuteness, in which a difference in the mode of aggregation is the only circumstance that constitutes a body solid, liquid, or gaseous. There are moreover strong reasons for believing, as we have seen, that these particles are inert masses, incapable, from any inherent power of their own, of modifying themselves, and susceptible only of obeying causes from without; whether this want of choice and self-direction is, in fact, as observation seems to prove, a general and essential characteristic of matter, or whether we so regard it intellectually for the purpose merely of considering by themselves those properties which remain to matter, after it is deprived of this. Now, material particles being considered as in this inert state, there will hence arise, in the phenomena which their aggregation presents, certain necessary conditions, which are applicable to all bodies, independently of the chemical nature of their constituent parts, being the simple consequences of their materiality. Such are the general laws of equilibrium and motion, which are deduced indeed mathematically from the single property of inertia.

14. We have already used the words *rest*, *motion*, and *force*, as making a part of ordinary language. It now becomes necessary to fix their meaning with precision. We begin with defining the place in which the phenomena under consideration are supposed to

occur. In order to this, let us conceive of space without bounds, immaterial, immovable, and of which all the parts, similar among themselves, are capable of being penetrated by matter without opposing the smallest resistance. Whether space in this sense exist in nature or not, is of little consequence; the definition presents to us merely an abstract extension. Now imagine in this space the particles of which we have been speaking, the material elements of bodies; and let us first consider with respect to them the mere circumstance of their existence. This simple fact will be capable of two distinct modifications; it may be that the same particle shall remain without change in its actual place, or that by the influence of external causes, it shall leave its place to pass to some other part of space. The first of these states constitutes *absolute rest*, and the second *motion*.

15. But we can conceive further, that two or several particles are displaced at the same time, and impressed with a common motion, preserving with regard to one another their respective positions. Then, if we consider them with reference to immovable space, they will actually be in absolute motion; but if we consider them simply in their mutual relations to one another, these will continue the same as if the whole group had remained at rest; and if there were upon one of these particles an intelligent being who should observe all the others, it would be impossible for him to decide from this observation alone, whether the whole system were in motion or not. The permanence of these relations in the midst of a common motion, is what we understand by *relative rest*. This will be the condition of a number of bodies placed in a boat and abandoned to the course of a smooth stream. This is indeed the condition of all the bodies about us, so long as they remain fixed to the same point of the terrestrial surface. They are at rest among themselves; but the earth, which turns daily on its axis, impresses upon them a common motion and at the same time bears them all together in its orbit round the sun, which perhaps in its turn carries the earth and the whole system of planets toward some distant constellation. Relative rest, therefore, is really the only kind of rest which can actually take place among the objects to which our attention is directed. It is at least all that we can ever be certain of observing.

16. We are hence led to make a similar distinction with respect to motion, and to separate the *absolute motions* of bodies, considered

with reference to immovable space, from the relative changes of position which may happen among them. These last therefore may be called *relative motions*, whether that body of the system to which they are referred, be itself in motion or at rest. The changes of place, for example, among the heavenly bodies, which we observe from the surface of the earth, are not absolute but relative motions, because the earth to which we refer them, as a fixed centre, has actually a motion of rotation on its axis and a progressive motion about the sun. Even when by calculation we have inferred from these observations the actual motions of the heavenly bodies as they would appear, if seen from the sun, we cannot affirm positively that these are absolute motions, since it may be that the sun and the whole planetary system have a common motion in space.

17. According to the idea of inertia which we derive from experience, we must regard the state of motion and that of rest, as simple accidents of matter, which it is incapable of imparting to itself, and which it can only receive from without, and which, once received, it cannot alter. When therefore we see a body passing from one of these states to the other, we must regard this change as produced and determined by the action of external causes. These causes, whatever they may be, are denominated *forces*. Nature presents us with an infinite number of them which are at least in appearance of different kinds. Such are the forces produced by the muscles and organs of living animals, the exercise of which, for the most part, depends solely on the will. Such are also the forces of physical agents, as the expansion of bodies by heat, and their contraction by cold, &c. There are moreover others which seem to be inherent in certain bodies, as the attraction of the magnet for iron, and that which is manifested among electrified bodies.

From the very nature of matter as thus presented to our consideration, it will be seen that *a body once put into a state of motion or rest, by any cause whatever, must continue in that state for ever, if no new cause is made to act upon it.* If it cannot give itself motion when at rest, it cannot stop itself when in motion, for this would be equivalent to giving itself motion in the opposite direction; neither can it change its velocity or direction, for this would equally imply a new force. Thus, *motion is naturally equal or uniform and rectilinear.*

18. When several forces are applied at the same time to a body, they are mutually modified by the connexion which exists among the different parts of the body, and which prevents each from taking the motion which the force exerted upon it tends to produce. If these forces happen entirely to destroy each other, so that the body remains at rest, we say that the forces are in *equilibrium*, or that the body is in equilibrium, under the action of these forces.

19. *Mechanics* is the science which treats of the equilibrium and motion of bodies. That part, the object of which is to discover the conditions of equilibrium, is called *statics*.* We give the name of *dynamics* † to the other part, which has for its object to determine the motion which a body takes, when the forces applied to it are not in equilibrium. The general laws of statics and dynamics are applicable to fluids; but, on account of the peculiar difficulty attending the consideration of this class of bodies, we are accustomed to treat them separately. That part of the mechanics of fluids which relates to their equilibrium is called *hydrostatics*, ‡ and that which comprehends their motions, *hydrodynamics*. §

20. In our inquiries on these subjects, we first proceed upon the supposition that there are no other bodies, and no other forces, in nature, except those under consideration. Thus all bodies are supposed to be destitute of weight, and free from friction, resistance, and obstructions of every kind. Regard is afterwards had to these causes; but to estimate their effects, it is necessary to begin by investigating each point separately.

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* From ἵστημι, *I stand*.
† From δύναμις, *power*.
‡ From ὕδωρ, *water*, and ἵστημι.
§ From ὕδωρ and δύναμις.

STATICS.

Chap. I

Of Uniform Motion.

21. A body is said to have a *uniform* motion when it passes continually over the same space in the same time.

In order to compare the motions of two bodies which move uniformly, it is necessary to consider the space which each describes in the same determinate time, as one minute, one second, &c. This space is what is called the *velocity* of the body.

22. The velocity of a body therefore is, properly speaking, only the space which this body is capable of describing uniformly in the interval of time which we take for unity.

Thus in the uniform motion of two bodies, the time being reckoned in seconds, if one passes over five feet in a second, and the other six feet in a second, we say that the velocity of the first is five feet, and that of the second six feet.

23. But if, the second being always taken as the unit of time, I am told that a body passes over 100 feet in 5 seconds, 100 feet does not express the velocity, since this space is not that which answers to the unit of time, a second; but it will be perceived, that in each second it would pass over a fifth part of this 100 feet, or 20 feet; that is, in order to find the velocity, I divide the number 100, the parts of the space passed over, by 5, the number of units in the elapsed time. Hence universally, *the velocity is equal to the space divided by the time*; for it is clear, that if we divide the whole space into as many equal parts, as there are units in the time elapsed, each part will be the space described during this unit of time, and will consequently be the velocity according to our definition. Thus

calling v the velocity and s the space passed over in any portion of time denoted by t, we shall have

$$v = \frac{s}{t};$$

this is one of the fundamental principles of mechanics.

24. The equation $v = \frac{s}{t}$ gives not only the measure of the velocity, but also that of the time and space. Indeed if we consider t and s as unknown quantities successively, we shall have, by the common rules of algebra,

$$t = \frac{s}{v}$$

and

$$s = v\,t;$$

Thus, *to find the time, we divide the space by the velocity; and, to find the space, we multiply the velocity by the time.*

If, for example, it is asked what time is required to describe 200 feet, when the body in question has a uniform velocity of 5 feet in a second; it is evident that it would require as many seconds as there are 5 feet in 200 feet; that is, we should have the time sought, or the number of seconds, by dividing the space 200 by the velocity 5; we shall find for the answer 40 seconds; or, in other words, a number of seconds equal to the quotient arising from dividing the space by the time.

In like manner, if it is asked what space would be described in 20 seconds by a body moving with a constant velocity of 5 feet in a second; it is manifest that it would describe 20 times five feet; that is, it is necessary in this case to multiply the velocity by the time.

Thus, although we have here employed algebraic characters, it is not because they are necessary to the investigation of these fundamental truths, but because, by means of them, the propositions, and their dependence, the one upon the other, are more concisely expressed, and more easily remembered. Indeed it will be seen by the above example, that the first principle, expressed algebraically, being once fixed in the mind, the two others are readily deduced from it by the most familiar rules.

25. It will be easy now to compare the uniform motions of two, or of a greater number of bodies. If it is asked, for example,

what is the ratio of the velocities of two bodies which describe the known spaces s, s', in the times t, t' respectively; by calling v, v', the velocities of these two bodies respectively, we shall have

$$v = \frac{s}{t}, \text{ and } v' = \frac{s'}{t'},$$

whence

$$v : v' :: \frac{s}{t} : \frac{s'}{t'};$$

that is, *the velocities are as the spaces divided by the times.*

In a word, if it is proposed to compare the velocities, the spaces, or the times, the principle above laid down, will give the expression for each of these particulars with respect to each body; we have therefore only to compare together these expressions. For example, if we would compare the spaces, the fundamental proposition $v = \frac{s}{t}$, gives $s = v\,t$; we have in like manner for the second body $s' = v'\,t'$; whence

$$s : s' :: v\,t : v'\,t',$$

that is, *the spaces are as the velocities multiplied by the times.*

26. Of these three things, namely, the space, time, and velocity, if we would compare two together, when the third is the same for each body, we have only to deduce from the same fundamental theorem, the expression for this third particular, with respect to each body, and to put these two expressions equal to each other. If, for example, we would know the ratio of the spaces when the velocities are the same, we should have

$$v = \frac{s}{t}, \text{ and } v' = \frac{s'}{t'}; \qquad 23.$$

whence, since by supposition $v = v'$, we have $\frac{s}{t} = \frac{s'}{t'}$, and accordingly

$$s : s' :: t : t';$$

that is, *the velocities being equal, the spaces are as the times.* It will be found in like manner that, *the times being equal, the spaces are as the velocities*; and that, *the spaces being equal, the velocities must be inversely as the times.* Indeed we have in this last case $s = v\,t$, and $s' = v'\,t'$; from which we obtain, when $s = s'$,

$$v : v' :: t' : t.$$

Thus the single proposition $v = \frac{s}{t}$ furnishes the means of comparing all the circumstances of uniform motion.

Of Forces and the Quantity of Motion.

27. The sum of the material parts of which a body consists, is called its *mass;* but in the use we shall make of the word is to be understood the number of material parts of which the body is composed.

Force, as we have said, is the cause which either moves or tends to move a body.

As forces interest us only by their effects, it is by the effects of which they are capable, that we are to measure them. Now the effect of a force is to cause in each particle of a body a certain velocity. Accordingly, if all the parts receive the same velocity as is here supposed, the effect of the moving cause has for its measure the velocity multiplied by the number of material parts contained in the body, that is, by the mass. Therefore, *a force is measured by the velocity, which it is capable of impressing upon a known mass, multiplied by this mass.*

28. The product of the mass of a body by its velocity is called *the quantity of motion* of this body. *Forces are therefore measured by the quantities of motion which they are capable of producing respectively.* Thus, if we designate the above product by p, the mass by m, and the velocity by v, we shall have $p = m\,v$. This equation gives $v = \frac{p}{m}$, and $m = \frac{p}{v}$; from which it will be seen that,

1. *The moving force of a body and its mass being known, we shall find the velocity by dividing the moving force by the mass;*

2. *The moving force and the velocity being known, we shall find the mass belonging to this velocity and moving force, by dividing the moving force by the velocity;*

3. *If the moving forces are equal, the velocities are inversely as the masses.*

The truth of these propositions is easily shown by putting successively the value of m equal to m', that of v equal to v', and that

of p equal to p'; the equations thus obtained, reduced and converted into proportions, form the several propositions above stated.

Remark.

29. The mass, or number of material parts of a body, depends upon its bulk or volume, and what is called its *density*, that is, the greater or less degree of closeness or proximity among its particles. As all bodies have more or less of void space within them, their quantities of matter are not proportional to their bulks; since, under the same bulk, the quantity of matter is greater according as the parts are more crowded and compressed together. A body is said to be more *dense* than another, when under the same bulk it has more matter; and, on the other hand, to be more *rare* than another, when under the same bulk it has less matter.

Accordingly, by means of the density of a body, we are able, when the bulk is known, to judge of the number of material parts which compose it; so that the density may be considered as representing the number of material parts in a given bulk. When we say that gold is 19 times as dense as water, we mean that gold contains 19 times as many parts in the same space.

30. By considering density as expressing the number of material parts of a determinate bulk, taken as the *unit of bulk*, it is evident that in order to find the mass, or total number of material parts, of a body whose bulk is known, we should simply multiply the density by the bulk. If, for example, the density of a cubic inch of gold be represented by 19, the quantity of matter contained in 10 cubic inches would be 10 times 19. Thus, designating the mass by m, the bulk by b, and the density by D, we shall have

$$m = b \times \text{D}.$$

It will hence be easy to compare together the masses, the bulks, and the densities of bodies.

Moreover, as the particles of matter, of whatever kind, tend by the force of gravity to move with the same velocity, or exert the same power, the combined action arising from this cause will be proportional to the number of particles; that is, the weight of a body is as its density, other things being the same. The relative weights of the different kinds of matter under equal bulks, are called the

specific gravities of the bodies respectively, the weight of pure water, in a vacuum, at a particular temperature, being taken as the unit. Thus, if a cubic inch, a cubic foot, &c., of gold be 19 times heavier than the same bulk of water, under the same circumstances, as to temperature, &c., the specific gravity of gold is said to be 19.

Of Equilibrium between Forces directly opposite.

31. We shall represent forces, as we have said, by their effects, that is, by the quantities of motion which they are capable of producing respectively, in a determinate mass. But, not to embrace too many objects at once, we shall consider each mass or body, as reduced to a single point, at which we suppose the same quantity of matter as in the body of which it takes the place. We shall see hereafter that there is in fact in every body a point through which motion is transmitted, as if the whole mass were concentrated there. We shall, moreover, unless the contrary is expressly stated, consider bodies as composed of particles absolutely hard, and connected together in such a manner as not to admit of any change in their respective situations by the action of any force whatever.

32. This being premised, let us suppose two bodies m, n, to be put in motion, the first from A toward C, with a velocity u, the second from C toward A with a velocity v. When these bodies come to meet, they will be in equilibrium, if the quantity of motion in m is equal to the quantity of motion in n; that is, if $m\ u$ is equal to $n\ v$. Fig. 1.

Indeed it is evident, that if m is equal to n, and the velocity u is equal to v, there must be an equilibrium; for in this case, whatever reason there may be for supposing m to prevail over n, might also be given for supposing n to prevail over m, since they are by hypothesis in all respects equal.

33. Let us suppose now, that m is double of n, but that v, at the same time, is double of u, that is, that n passes over two feet, for example, in a second, while m passes over one foot in a second. It is clear that we may consider m as composed of two masses equal each to n; and that, at the instant of meeting, we may represent the body n as having a velocity of one foot in a second, to which is added, at the same instant, another velocity of one foot in a second. We may then conceive, that, in meeting, the mass n expends one of

its velocities against a portion of the mass m equal to itself, and its other velocity against the remaining portion of m of the same magnitude.

If now, instead of supposing the masses m and n in the ratio of 2 to 1, and their velocities in the ratio of 1 to 2, we suppose them in any other ratio, it is evident that we may always conceive the greater mass as decomposed into a certain number of portions equal each to the smaller, and of which each shall destroy, in the smaller, a velocity equal to its own. We may therefore consider the following proposition as established.

Two bodies which act directly against each other in the same straight line, are in equilibrium when their quantities of motion are equal; that is, when the product of the mass of the one into the velocity with which it moves, or tends to move, is equal to the product of the mass of the other into its actual or virtual velocity.

This proposition is to be regarded as general, whether the two bodies move freely and directly the one against the other, or whether they act against each other by the intervention of a rod inflexible and without mass, or whether they are considered as pulling in opposite directions by means of a thread $m\ n$ incapable of being extended. And reciprocally, if two bodies are in equilibrium, we may conclude that their motions are directly opposite, and that their quantities of motion are equal.

34. We infer, moreover, that if three or a greater number of bodies m, n, o, &c., moving, or tending to move, in the same Fig. 2.
straight line, with velocities u, v, w, &c., are in equilibrium, the sum of the quantities of motion of those which act in one direction is equal to the sum of the quantities of motion of those which act in the opposite direction. For, they being in equilibrium, we may always suppose that, m and n acting in the same direction, n destroys a part of the motion of o, and that m destroys the remaining part. Now if we represent by x the velocity that o loses by the action of n, we shall have $o\ x$, for the quantity of motion destroyed by the action of n; we have accordingly

$$n\ v = o\ x.$$

The body m, therefore, will have only to destroy in o the remain-

ing quantity, namely, $o\,w - o\,x$; we have consequently

$$m\,u = o\,w - o\,x;$$

or since

$$o\,x = n\,v,$$

$$m\,u = o\,w - n\,v,$$

that is,

$$m\,u + n\,v = o\,w.$$

Of Compound Motion.

35. We still consider the masses to which the forces under consideration are applied, as concentrated each in a point.

We call *compound motion* that which takes place in a body, when urged at the same time by two or more forces having any given direction with each other.

Fig. 3. If a body *m* moving in the line *CB*, receive, upon arriving at the point *A*, an impulse in the direction *AD*, perpendicular to *CB*, this impulse can produce no other effect, except that of removing the body from *CB*. It can neither augment nor diminish the velocity with which, at the time of receiving the impulse, it was departing from *AD*. Indeed, since *AD* is perpendicular to *CB*, there is no reason why a force acting in the direction *AD* should produce an effect to the right, rather than to the left, of this line, and as it cannot act in both these directions at once, it can have no influence either way.

The same reasoning will hold true, if we suppose that the body *m*, moving in the line *AD*, receives, upon arriving at *A*, an impulse in the direction *AB*. This impulse will neither add to nor take from the velocity with which the body *m* was departing from *AB*.

Fig. 4. 36. *If two forces* p *and* q, *the directions of which are at right angles to each other, act at the same instant upon a body* m, *and the force* q *is such as by its sole and instantaneous action to cause the body to pass over* AB *in a determinate time, as one second, and the force* p *is such as to cause the body to pass over* AD *in the same time, we say that by the joint action of the two forces,* q *and* p, *the body* m *will in the same time pass over the diagonal* AE, *of the parallelogram* DABE, *which has for its sides these same lines,* AB, AD.

Since the two forces act at the same instant upon the given body, we may suppose it moving in the line *AD*, and that at the in-

stant of its arriving at the point A, it receives the force q in a direction perpendicular to AD. Now, according to article 35, the force q can neither increase nor diminish the velocity with which it was at this moment departing from AB; if, therefore, through the point D we draw DE parallel to AB, the body must at the end of a second be somewhere in the line DE, all parts of which are equally distant from AB.

The same reasoning may be adopted with regard to the force q, by which it will be seen, that if, through the point B, we draw BE parallel to AD, the body must at the end of a second be somewhere in BE. But there is only the point E which is at the same time in DE and BE; therefore at the end of a second the body will be in E.

It is also evident, that whatever course the body takes, by the instantaneous action of the forces, this course must be a straight line, since, from the instant that the forces are exerted, the body 17.
is abandoned to itself, and there is no cause to incline it one way rather than another. Accordingly, as this body passes through A and E, and without any thing to change its direction, the course must be AE, that is, the diagonal of the parallelogram $DABE$.

We will add moreover, that the body describes AE with a uniform motion, since, after the joint action of the two forces, it is left equally without any cause to alter its rate of moving. 17.

37. Since the two forces p and q, acting simultaneously upon the body m, have no other effect than to make it describe the diagonal AE, we infer, that, instead of two forces whose directions are at right angles to each other, we may always substitute a single one, provided that this single one is such as to cause the body to describe the diagonal of a right-angled parallelogram, the sides of which would be described in the same time, each separately, by the action of the force of which it represents the direction.

The single force AE, which results from the action of the two forces AB, AD, is called the *resultant* of these two forces. As the lines AB, AD, represent the effects which the forces q and p are singly capable of producing, and AE the effect which they are able to produce conjointly, we may regard AB, AD, AE, as representing these forces themselves.

We may thus consider any single force AE, as being the result of two other forces AB, AD, the directions of which are at right

angles to each other, provided that, the first being represented by the diagonal AE, the others are represented by the sides AB, AD, of this same right-angled parallelogram. For the single force AE, therefore, we may substitute the two forces AB, AD, since these two will in fact only produce AE.

33. In general, *whatever be the angle formed by the directions*
Fig. 5,6. *of the two forces* p *and* q *which act at the same time upon a body* m, *this body will still describe the diagonal* AE, *of the parallelogram* DABE, *the sides of which represent, in the directions of the forces, the effects which they are separately capable of producing; and the body will describe this diagonal in the same time in which, by the action of either of the two forces, it would have described the side which represents this force.*

Through the point A let the line FAH be drawn perpendicular to the diagonal AE, and through the points D and B, let DF, BH, be drawn parallel, and DG, BI, perpendicular to the diagonal AE. Instead of the force p, represented by AD, the diagonal of the rectangular parallelogram $FAGD$, we may take the two forces AF,
37. AG. For the same reason, instead of the force q, represented by the diagonal AB, of the rectangular parallelogram $AHBI$, we may take the two forces AH, AI. We may therefore, instead of the two forces p and q, substitute the four forces AF, AG, AH, AI; and these cannot but have the same resultant as the two forces p and q. Now of these four forces, the two AH, AF, contribute nothing to the resultant, because they act in opposite directions, and are equal to each other. Indeed it will be readily seen, that, from the na-
32. ture of a parallelogram, the two triangles DGA, EIB, are equal; therefore $DG = BI$, and consequently $AF = AH$.

As to the two forces AI, AG (*fig.* 5.), since they are exerted according to the same line, and are directed the same way, the result must be the sum of the two effects AG, AI; and in *fig.* 6, since AI, AG, are exerted according to the same line, and are opposed the one to the other, the result must be the difference of the two effects AG, AI. But as the triangle EIB is equal to DGA, we shall have (*fig.* 5.)

$$AI + AG = AI + EI = AE;$$

and (*fig.* 6.)

$$AI - AG = AI - EI = AE.$$

We conclude, therefore, that the four forces AF, AH, AG, AI,

and consequently the two forces AD, AB, have no other effect, than the force AE, represented by the diagonal of the parallelogram $DABE$, of which the two sides AB, AD, denote the forces q, p. This proposition is known by the name of the *parallelogram of forces.*

39. We have, in what precedes, represented the two forces p, q, by the lines AD, AB, which they are capable of making the Fig. 4,5,6.
body m describe in the same time, that is, by the velocities which they would communicate; although, according to what we have said, the true measure of any force is the quantity of motion that 28.
it is capable of producing. But as the quantities of motion are in the ratio of the velocities, when the mass is the same, as is the fact 29.
in the present case; we may always, as we have now done, take the velocities AD, AB, as representing the two forces.

But if, instead of having immediately the velocities which the two forces p, q, are capable of giving to the body m, we had the quantities of motion which they would produce in known masses, we should take AD, AB, in the ratio of these quantities of motion. If, for example, I know the forces p, q, only by this circumstance, that the force p is capable of giving a known velocity u, to a known mass n; and that the force q is capable of giving a velocity v to a known mass o; I should take

$$AD : AB :: nu : ov.$$

For, according to what has been shown, AD, AB, are to be taken in the ratio of the velocities which they are capable of giving to the body m. Now the first being capable of producing the quantity of motion nu, is capable of giving to the body m the velocity $\frac{nu}{m}$. For 28.
the same reason the second, or the force q, is capable of giving to the body m, the velocity $\frac{ov}{m}$. The lines AD, AB, are consequently to be taken according to the following proportion;

$$AD : AB :: \frac{nu}{m} : \frac{ov}{m};$$

but

$$\frac{nu}{m} : \frac{ov}{m} :: nu : ov.$$

We see therefore, that AD, AB, must be in the ratio of the quantities of motion nu, ov, which are the measures respectively of the forces p, q.

What has now been remarked, will be found useful in comparing the effects of different forces applied to different bodies.

38. The general proposition above demonstrated is of the greatest importance, as almost every thing we have to offer, consists in an application of it.

40. From what has been said, it will be seen that it is immaterial whether we regard a body as urged by the combined action
Fig. 5,6. of the two forces AB, AD, which make with each other any assumed angle, or whether we regard it as urged by the single action of a force represented by the diagonal AE.

And reciprocally, it amounts to the same thing, whether we consider a body as urged by a single force AE, or by two forces represented by the two sides of a parallelogram of which the single force AE is the diagonal. Let a body, for example, be supposed to pass from A to E by a uniform motion in one second, or let it be supposed to move through AB at such a rate as to describe it in one second, while in the same time this line is carried parallel to itself along AE; in this case, as in the former, the body will merely describe the line AE.

41. The two forces AB, AD, meeting at the point A, are ne-
Geom. 321. cessarily in the same plane. Since, therefore, they have for their resultant the diagonal AE, which is in the plane of the parallelogram, we may infer generally that any two forces which unite in the same point are always in the same plane with their resultant.

Of the Composition and Decomposition of Forces.

42. Not only is it possible, by the principle above established, to reduce two concurring forces to one, and to decompose one into two others; but we can in general reduce to a single force, as many other forces as we please, when they are in the same plane, or when they unite at the same point; and reciprocally, we can decompose one or several forces into as many other forces as we please.

43. But before we proceed to explain this, we must observe
Fig. 7. that when a force p acts upon a body either by pushing or by drawing it, it is of no consequence at what point of the direction of this force we suppose the action to be applied. For example, let the force p be exerted upon the body m by means of a rod

inflexible and without mass, or by a thread inextensible and without mass, it is the same thing, whether the force p be applied at the point B, or at the point C, or whether it be of such a nature as to admit of being exerted at any point D, on the other side of the body. So long as its action is employed in the same direction, the effect will be the same. Distance can have no influence, except so far as the action of the power transmits itself by the aid of some instrument, as a lever or a cord, the matter of which would partake of the action of the power, all which instruments we at present leave out of consideration.

Thus, if two forces p and q, exerted in the same plane, according Fig. 8. to the lines EC, DB, draw or push a body by the two points E, D, this body is urged in the same manner as it would be, if the two forces were both employed at their point of meeting A, the directions being supposed to remain unchanged.

This being premised, we proceed to the consideration of the composition and decomposition of forces.

44. Let there be four forces, p, q, r, ϖ, directed in the manner Fig. 9. represented in the figure,* and all in the same plane. Let us imagine the direction of the force p, prolonged till it meets that of the force q in the point A; AD, AE, being supposed to be the spaces that the forces p, q, can respectively cause the same body to describe, in a determinate time, as one second; if we form the parallelogram $AEID$, the diagonal AI will represent the resulting effort of p and q, and may consequently take the place of these two forces.

Let us conceive now that AI prolonged, meets in B the direction of the force r, and having taken BL equal to AI, if we take BF for the space that the force r is capable of making the same body describe in a second; the force AI being supposed to be applied at B, since this force is represented by $BL = AI$, from its action combined with the force $r = BF$, there will result a single force represented by the diagonal BG, of the parallelogram $BLGF$. This force, therefore, will take the place of the forces r and AI, that is, it will take the place of the three forces r, q, and p.

Lastly, let us imagine that BG prolonged, meets in C the direction of the force ϖ, and let us make $CK = BG$. Let CH

* The direction of a force is indicated by the figure of an arrow.

represent the space which the force ϖ is capable of making the same given body describe in a second; then by supposing the force $BG = CK$, applied at C in the direction CG, from the union of this force with the force ϖ there will result a single effort represented by the diagonal CN, of the parallelogram $CHNK$. This force, therefore, will take the place of the forces ϖ and CK, or of ϖ and BG; it will consequently take the place of the four forces p, q, r, ϖ, and is accordingly the resultant of these four forces.

It is evident, therefore, that any number of forces, when exerted in the same plane, may be reduced to a single force, and the manner in which this may be done is also manifest.

45. It will be seen moreover from the above example, how we may always substitute for a single force as many others as we please, and what are the requisite conditions for effecting this.

Instead of the single force BG, for example, we may, by forming the parallelogram $BLGF$, of which BG is the diagonal, take the two forces represented by BF, BL; and as we may suppose each of these two forces applied at any such point of their directions respectively as we choose, we can transfer BL to AI, (the point A being at any assumed distance from B,) and form upon AI, another parallelogram $AEID$; then for the force AI may be substituted two forces represented by AE, AD; so that for the single force BG, we shall have the three forces BF, AE, AD, the effect of which will be equivalent to that of BG.

46. We remark here, that, since there is no other condition required for determining the forces AD, AE, except that they be expressed by the sides AD, AE, of the parallelogram $ADIE$ of which AI is the diagonal, which condition may be fulfilled in an infinite number of ways, whether the parallelogram $ADIE$ be in the same plane with the parallelogram $FBLG$ or in any other plane, we can decompose any force whatever BG into as many others as we please, and which shall be in such planes as we please. We shall see hereafter the use that may be made of this method of compounding and resolving forces.

47. From what is above said, it will be perceived that we can require certain forces to pass through certain given points, and even to be of certain determinate magnitudes, to be parallel to certain given lines, in a word, to satisfy certain given conditions. For ex-

ample, if we had a force represented by the line AB, and we would substitute two others, of which one should pass through the point D, (in a direction parallel to a line LX, whose position is given,) and which at the same time should be of a certain magnitude LK, that is, such as would cause a given body to describe LK, in the same time in which the force represented by AB would cause this same body to describe the line AB; the principles above established will enable us to solve the problem. Fig. 10.

Through the point D, we draw IC parallel to LX, and meeting AB produced in some point I; we take $IC = LK$, and $IE = AB$; then joining CE, we draw through the point I the line IH parallel to CE, and through E the line HE parallel to IC; IC will be the force required, and IH will be the force which, combined with IC, would take the place of IE or of AB.

The solution we have given will always be applicable, except when the line LX is parallel to AB, and we shall see soon what is to be done in this case.

48. We remark further, that, the two component forces p, q, being represented by the two sides AD, AB, of the parallelogram $DABE$, their resultant must necessarily be represented by the diagonal AE of the same parallelogram; by calling ϱ the resultant, we shall have Fig. 5, 6.

$$p : \varrho :: AD : AE,$$
$$q : \varrho :: AB : AE;$$

that is,

$$p : q : \varrho :: AD : AB : AE,$$
$$:: BE : AB : AE.$$

Now in the triangle ABE, we have

$$BE : AB : AE :: \sin BAE : \sin BEA : \sin ABE.$$ Trig. 32.

But on account of the parallels BE, AD, the angle $BEA = DAE$, Geom. 67.
and the angles ABE, BAD, being supplements to each other, Geom. 64.
$\sin ABE = \sin BAD$; hence Trig. 13.

$$BE : AB : AE :: \sin BAE : \sin DAE : \sin BAD;$$

and consequently

$$p : q : \varrho :: \sin BAE : \sin DAE : \sin BAD;$$

from which it will be seen, that if we suppose the force p expressed by $\sin BAE$, the force q will be denoted by $\sin DAE$, and the force ϱ by $\sin BAD$; that is, *the two component forces and the re-*

sultant may be represented each by the sine of the angle comprehended between the directions of the two others.

In representing forces, therefore, we may employ indifferently either the lines taken in the directions of these forces, or the sines of the angles comprehended between these directions, provided we take for each the sine of the angle comprehended between the directions of the two others.

This last method of expressing forces has its peculiar advantages, as we shall see in what follows.

Fig. 11, 12. 49. If from the point A as a centre, and with any radius AC, we describe an arc of a circle HCG, meeting in G and H the directions of the forces p, q, and let fall from the point C upon AD, AB, the perpendiculars CF, CI, and from the point H upon AD the perpendicular HL, it will be readily seen that CF, CI, HL, are the sines of the angles DAE, BAE, BAD, respectively; we have accordingly

$$p : q : \varrho :: CI : CF : HL.$$

Fig. 13, 14. 50. Let us suppose now, that while the directions of the two forces p, q pass through the two fixed points K, N, their point of meeting A is removed further and further; it is evident that the sines of the angles BAE, DAE, BAD, will approach more and more to a coincidence with the arcs CH, CG, HG; if therefore the point A is removed to an infinite distance from the fixed points K, N, CF, CI, HL, will coincide with the arc HG, which in this case becomes a straight line perpendicular to the two lines AK, AN, which are then parallel to each other and to

Geom. 12. the line AE; and, since we have always

$$p : q : \varrho :: CI : CF : HL,$$
$$HL = CI + CF \ (fig.\ 13).$$
$$HL = CI - CF \ (fig.\ 14).$$

Fig. 15, 16. We conclude, therefore, that when two forces p, q, are exerted in parallel directions,

1. *That their resultant is in a direction constituting another parallel;*

2. *That if we draw a line* FI *perpendicular to these directions, each of the forces will be represented by the part of this perpendicular comprehended between the directions of the two others;*

3. *That the resultant is equal to the sum of the two components, when these act the same way, and to their difference, when their action is opposed the one to the other.*

51. Since we have

$$p : q : \varrho :: EI : EF : FI,$$

we have

$$p : q :: EI : EF, \quad \text{and} \quad p : \varrho :: EI : FI;$$

that is, of two parallel component forces and their resultant, either two are to each other reciprocally, as the two perpendiculars let fall upon their directions respectively from the same point in the direction of the third.

52. If we draw arbitrarily any line ABC, we shall have Geom. 198.

$$BC : AB : AC :: EI : EF : FI,$$

and consequently

$$p : q : \varrho :: BC : AB : AC;$$

that is, *if a straight line be drawn at pleasure, cutting the directions of two parallel forces and their resultant, each of these forces will be represented by that part of the straight line which is comprehended between the directions of the two others.*

53. It will hence be readily perceived how we ought to proceed in order to find the resultant of several parallel forces; and reciprocally, how we can substitute for a single force, any number whatever of parallel forces.

If, for example, it were proposed to reduce to a single force the two parallel forces p, q, which act the same way; any straight Fig. 15.
line ABC being drawn; as the resultant ϱ is equal to $p + q$, it is 50.
only necessary to find the point B through which this resultant must pass. Now we have 50.

$$p : \varrho :: BC : AC,$$

that is,

$$p : p + q :: BC : AC.$$

We have therefore only to take between the two points A, C, a point B such that BC shall be equal to $\frac{p \times AC}{p + q}$. Geom. 237.

If the two parallel forces are opposed to each other, the resultant Fig. 16.
will be equal to their difference $p - q$, or $q - p$. Suppose p 50.
greater than q. Having drawn the line AC at pleasure, it will be

necessary to prolong AC beyond A, with respect to C, by a quantity AB, such that we shall have

52. $$p : \varrho :: BC : AC$$

or

$$p : p - q :: BC : AC;$$

in other words, it is necessary to take BC equal to $\frac{p \times AC}{p - q}$.

If q is greater than p, the point B will be in AC produced beyond C with respect to A.

Fig. 17. 54. If we had a third force r, we should first find the resultant ϱ' of the two forces p, q, and then seek the resultant ϱ of the two forces ϱ' and r, as if there were only these two; that is, we should proceed in precisely the same manner as we have done in the preceding article.

Fig. 15, 16. 55. Hence, reciprocally, if we would decompose any force ϱ into two others parallel to it, we should take arbitrarily a line AF parallel to the direction of ϱ; and having assumed this line as the direction of one of the components, we take arbitrarily for the value of this component any quantity p smaller than ϱ, if it is proposed that the two components should act on opposite sides of the force ϱ; the second component q must in this case be equal to $\varrho - p$; and in order to find its position, it is only necessary, having drawn any straight line CBA, to take in AB produced the part BC, such as to give the proportion

$$q : p :: AB : BC;$$

then through the point C we draw IC, parallel to EB, and this will be the direction of the force q.

But if the two component forces are required to be on the same side (in which case they will be directed opposite ways), then we take for p any quantity, whether greater or less than ϱ, and if it be greater it will be directed the same way with ϱ, and if less, it will have a contrary direction with respect to ϱ. Having drawn a line AF parallel to EB, as the direction of p, we take upon any assumed line BAC the point C, such as will give

$$p - \varrho \text{ or } \varrho - p : \varrho :: AB : AC;$$

and C will be the point through which the force q must pass parallel to the given force ϱ; and the point C will be beyond A with

respect to B, when p is greater than ϱ; and it will be between A and B when p is less than ϱ.

56. Since what we have now said of the force ϱ with respect to the components p, q, may evidently be applied to each of these latter forces, it will be seen how we may substitute for any single force, as many others as we please, the directions of which are parallel.

CHAP. VII.

Of Moments and their Use in the Composition and Decomposition of Forces.

57. The propositions we have established, are sufficient for the composition and decomposition of forces, whatever be their magnitudes and directions, provided they act in the same plane. But the different kinds of motion which we have to consider, require more simple and more expeditious means for determining the resultant of forces, and its direction.

58. *If from any point* F *taken in the plane of any parallelogram* ABCD, *we let fall upon the contiguous sides* AB, AD, *and the diagonal* AC, *the perpendiculars* FE, FH, FG, *the sum of the products of the two contiguous sides by the perpendiculars respectively let fall upon them, will be equal to the product of the diagonal by its perpendicular, when the point* F *is neither in the angle* BAD, *nor in the vertical angle* KAL. *If, on the contrary, the point* F *is either in the angle* BAD, *or in the vertical angle* KAL, *the difference of the products of the two contiguous sides by their respective perpendiculars will be equal to the product of the diagonal by the perpendicular let fall upon it.* Fig. 18, 19. Fig. 18. Fig. 19.

Produce the side BC till it meets in I the perpendicular FH, and join FA, FB, FC, FD. The triangle

$$FAC = FAB + ABC + FBC = FAB + ADC + FBC.$$ Fig. 18.

Now

1. The triangle $FAC = \dfrac{AC \times FG}{2}$. Geom. 176.

2. The triangle $FAB = \dfrac{AB \times FE}{2}$.

3. The triangle ADC having AD for its base, and IH for its altitude,

$$ADC = \frac{AD \times IH}{2}.$$

4. The triangle $FBC = \frac{BC \times FI}{2} = \frac{AD \times FI}{2}$.

Whence

$$\frac{AC \times FG}{2} = \frac{AB \times FE}{2} + \frac{AD \times IH}{2} + \frac{AD \times FI}{2}.$$

Now $IH + FI = FH$; therefore, by doubling the whole, we have

$$AC \times FG = AB \times FE + AD \times FH.$$

Fig. 19. With respect to the triangle FAC, we have

$$FAC = ABC - FAB - FBC = ADC - FAB - FBC,$$

that is,

$$\frac{AC \times FG}{2} = \frac{AD \times IH}{2} - \frac{AB \times FE}{2} - \frac{BC \times FI}{2};$$

or, since $BC = AD$, and $IH - FI = FH$, the whole being doubled,

$$AC \times FG = AD \times FH - AB \times FE.$$

59. Since we have before shown that any two forces and their resultant may be represented by the sides and diagonal of a parallelogram, formed upon the directions of these forces, if p, q, be two forces, represented by the lines AB, AD, in which case their resultant ϱ would be represented by AC, any point F being taken in the plane of these three forces without the angle BAD, and without the vertical angle KAL, we have always

$$\varrho \times FG = p \times FE + q \times FH;$$

and when the point F is taken in the angle BAD, or in the vertical angle KAL, we shall have in like manner

$$\varrho \times FG = q \times FH - p \times FE.$$

60. The product of a force by the distance of its direction from a fixed point is called the *moment* of this force. Thus $q \times FH$ is the moment of the force q; and $\varrho \times FG$ is the moment of the force ϱ.

61. As a force is estimated by its quantity of motion, that is, by the product of a determinate mass into the velocity which it is capable of giving to this mass, the moment of any force has for its measure the product of a mass, by its velocity, and by the distance of its direction from a fixed point.

62. If the perpendiculars FH, FG, FE, are considered as lines inflexible and without mass, connected together and fixed to

the point F, in such a manner as to admit only of their turning about this point; and we suppose that the forces p, q, and their resultant ϱ, are applied at the extremities E, H, G, we shall see that these three forces tend each to turn the system in the same Fig. 18.
direction about the point F; and that the two forces q, ϱ, tend to Fig. 19.
turn the system in a different direction from that in which the force p tends to turn it.

We infer, therefore, that *the moment of the resultant, taken with respect to any fixed point* F, *is always equal to the sum or to the difference of the moments of the two components, according as these components tend to turn the body or system in the same direction, or in opposite directions, about this fixed point.*

63. We conclude, moreover, that in general, *whatever be the number of forces* p, q, r, ϖ, *&c., and whatever their magnitudes and* Fig. 20.
directions, provided they act in the same plane, the moment of the resultant of all these forces, taken with respect to a fixed point F, *assumed at pleasure in this plane, will always be equal to the sum of the moments of the forces tending to turn the system in one direction about this point, minus the sum of the moments of those which tend to turn it in the opposite direction.*

Indeed, if we suppose that ϱ' is the resultant of the two forces Fig. 20.
p, q, ϱ'' that of ϱ' and r, and ϱ that of ϱ'' and ϖ; if we suppose, moreover, that μ represents the moment of ϱ', and μ' that of ϱ'', then by letting fall the perpendiculars FA, FE, FG, FD, FB, upon the components p, q, r, ϖ, and their resultant ϱ, we shall have

$$1.\ \mu = p \times FA + q \times FE,$$
$$2.\ \mu' = \mu - r \times FG,$$
$$3.\ \varrho \times FB = \mu' - \varpi \times FD;$$

adding therefore these three equations together, and suppressing those quantities that cancel each other in the two members, we shall have

$$\varrho \times FB = p \times FA + q \times FE - r \times FG - \varpi \times FD;$$

from which it will be seen, that the moments of the two forces r, ϖ, which tend to turn the system from right to left, are of a contrary sign to that of the forces p, q, which tend to turn it from left to right.

64. If the point F were exactly in the direction of the resultant, the moment of this force would be zero; but, since it is equal

to the sum of the moments of the forces which tend to turn the system in one direction, minus the sum of the forces tending to turn it in the opposite direction, we conclude that the difference of these two sums of moments, taken with respect to any point whatever in the direction of the resultant, is zero.

And reciprocally, *if the sum of the moments of the several forces which tend to turn a system about a given point, minus the sum of the moments of those which tend to turn it in the opposite direction about this same point, is zero; it must be inferred, that the resultant passes through this point.*

65. As these propositions hold true, whatever be the angles formed by the directions of the forces, they are applicable, when these angles are infinitely small, that is, when the directions of the forces are parallel.

66. We may thus derive a very simple method of obtaining the position and magnitude of the resultant of any number of forces, when they all act in the same plane.

Let us suppose, in the first place, that they are all parallel; and, not to make the problem more complicated than is necessary, let us suppose that there are only three forces; it will be easily inferred how we are to proceed in case of a greater number.

Fig. 21. Accordingly, let there be the three known forces p, q, r, the two first being directed the same way, and the third having a contrary direction. Having drawn arbitrarily any line $FABC$, perpendicularly to the directions $A\,p$, $B\,q$, &c., we will suppose that D is the point through which the resultant ϱ is to pass. Then, having taken at pleasure a point F in $FABC$, we shall have, according to
63. what has been demonstrated,

$$p \times FA + q \times FB - r \times FC = \varrho \times FD.$$

Now the distances FA, FB, FC, and the forces p, q, r, being known, it will be easy to deduce from the above equation, the value of the distance FD, through which the resultant would pass, if the value of this resultant ϱ were known. In order to find it, we take another point φ in AF produced, and by proceeding as above, we have

$$p \times \varphi A + q \times \varphi B - r \times \varphi C = \varrho \times \varphi D.$$

If from this second equation we subtract the first, recollecting that

$$\varphi A - FA = \varphi F, \quad \varphi B - FB = \varphi F,$$
$$\varphi C - FC = \varphi F, \quad \varphi D - FD = \varphi F,$$

we shall have

$$p \times \varphi F + q \times \varphi F - r \times \varphi \mathrm{F} = \varrho \times \varphi F;$$

that is, the whole being divided by φF,

$$p + q - r = \varrho.$$

If we examine the process now pursued, we shall see that it does not depend in any degree upon the number of forces, but that it is applicable, whatever this number may be. We must infer therefore, that the resultant of any number of parallel forces is equal to the sum of those which act in one direction minus the sum of those which act in the opposite direction.

If now in the equation

$$p \times FA + q \times FB - r \times FC = \varrho \times FD,$$

found above, we put for ϱ its value $p + q - r$, just obtained, we shall have

$$p \times FA + q \times FB - r \times FC = (p + q - r) \times FD,$$

from which we deduce

$$FD = \frac{p \times FA + q \times FB - r \times FC}{p + q - r};$$

or, bearing in mind, that the process by which we have arrived at this result, does not depend upon the number of forces employed, we infer, as a general conclusion, that *in order to determine at what distance from a given point the resultant of several parallel forces passes, from the sum of the moments of the forces which tend to turn the system in one direction, we must subtract the sum of the moments of the forces tending to turn it in the opposite direction, and divide the remainder by the sum of the forces which act in one direction, minus the sum of those which act in a contrary direction.**

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* We must take care not to confound the forces which act in opposite directions, with those which tend to turn the system in opposite directions. Two forces which act in opposite directions often tend to turn the system in the same direction. This depends upon the point to which the rotation, or the moments, is referred. The two forces q, r, for example, act in opposite directions, but Fig. 21. they both tend to turn the line BC in the same direction, about a

67. If the point F, assumed arbitrarily, should happen to be so taken as to fall in D, through which the resultant passes, the distance FD being zero, its value

$$\frac{p \times FA + q \times FB - r \times FC}{p + q - r},$$

since the force p tends to turn the system about the point D in a direction opposite to that in which the force q tends to turn it, becomes

$$\frac{-p \times DA + q \times DB - r \times DC}{p + q - r},$$

and is equal to zero; we have consequently

$$-p \times DA + q \times DB - r \times DC = o,$$

or

$$q \times DB = p \times DA + r \times DC.$$

44. Moreover, as the point F, taken arbitrarily, may be higher or lower, as we please, the point D has not been supposed to be in one point of the direction of the resultant rather than in another; it follows, therefore, that *the moments of several parallel forces, taken with respect to any point whatever in the direction of the resultant, are such, that the sum of the moments of the forces which tend to turn the system in one direction, is always equal to the sum of the moments of those which tend to turn it in the opposite direction.*

68. Therefore by taking with contrary signs the moments of the forces which tend to turn the system in opposite directions, and by taking also with contrary signs the forces which act in opposite directions, we may infer as a general conclusion;

1. *That the resultant of any number whatever of parallel forces is always equal to the sum of all these forces;*

2. *That this resultant, which is parallel to the component forces, passes through a series of points each of which has this property, that the sum of the moments, taken with respect to this point, is zero.*

The above propositions are of the greatest importance. We shall see soon with what facility they enable us to find the centre

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point taken between B and C; and if we consider the rotation with reference to the point F, the force q tends to turn FC in a direction opposite to that in which the force r tends to turn it.

of gravity of bodies. We proceed now to the consideration of forces the directions of which are inclined to each other.

69. Let there be any number of forces p, q, r, &c., all exert- Fig. 22.
ed in the same plane, let the force p, acting according to AE, be represented by AE, and let the force q, acting according to BG, be represented by BG, and the force r, acting according to CL, be represented by CL. Through a point F, taken arbitrarily in the plane of these forces, suppose two straight lines FC', FB'', making any angle with each other (and for the sake of greater simplicity, let this angle be a right angle); and let us imagine the forces p, q, r, or AE, BG, CL, decomposed each into two others, one of which shall be parallel to FC', and the other to FB'', and which will consequently be represented, each by the corresponding side of the parallelogram, the diagonal of which represents the 40.
given force.

It is clear from what has been said, that the forces AV, BH, 66, 68.
CK, being parallel, will have for their resultant a single force DN, parallel to AV, BH, &c., the value of which will be

$$AV + BH + CK,^*$$

and which will pass at a distance $D'D$ from FC', equal to the expression below, namely,

$$D'D = \frac{AV \times AA' + BH \times BB' + CK \times CC}{AV + BH + CK}.$$

In like manner, the forces AI, BR, CM, parallel to FB'', are reduced to a single one DO, parallel to AI, &c., and equal to $AI + BR - CM$, and which (by supposing that D is the point where the direction of this force meets that of the force ND) will pass at a distance $D''D$ from FB'', equal to the following expression, namely,

$$D''D = \frac{AI \times AA'' + BR \times BB'' - CM \times CC''}{AI + BR - CM}.$$

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* We must not lose sight of what was said art. 39. By the forces AE, BG, &c., we are to understand that the lines AE, BG, &c., are to each other as the quantities of motion capable of being produced by the forces p, q, &c., in the masses to which they are applied. It is to be observed, likewise, with respect to the forces AV, BH, &c., that we mean by them quantities of motion, which are to the quantities of motion represented by AE, BG, &c., as AV, BH, &c., are to AE, BG, &c., respectively.

This being supposed, the forces p, q, r, and their directions, (that is, the angles which they make with the known fixed lines FC', FB'', or with their parallels,) being considered as known, we know in each of the triangles AEI, BGR, CLK, the hypothenuse and the angles. It will accordingly be easy to determine the lines
Trig. 30. AI, BR, KL, or CM, and the lines IE or AV, RG or BH, and CK. We shall consequently know the values of the two resultants $AV + BH + CK$, and $AI + BR - CM$. Moreover, as we cannot but know the distances AA', AA'', BB', BB'', &c., since the position of the points A, B, &c., where the forces are applied are supposed to be given, we are acquainted with all the quantities which enter into the expression of the distances $D'D$, $D''D$. It will be easy, therefore, to determine the point D, where these two resultants meet. Accordingly, taking

$$DO = AI + BR - CM, \text{ and } DN = AV + BH + CK,$$

and forming the parallelogram $DNTO$, we shall have the diagonal DT for the resultant ϱ, of the two partial resultants, parallel to FC' and FB'', that is, for the resultant of all the proposed forces.

Of Parallel Forces which act in different Planes.

Fig. 23. 70. Let there be three forces p, q, r, directed according to the lines $A\,p$, $B\,q$, $C\,r$, parallel to each other, but situated in different planes.

Imagine a plane XZ to which the three straight lines $A\,p$, &c., are perpendicular, and another plane ZV to which they are parallel, and let A, B, C, be the three points where these lines meet the plane XZ.

Geom. The two forces p, r, are in the same plane, the intersection of
335,324. which with the plane XZ is the straight line AC. These two
50. forces may therefore be reduced to a single one ϱ', equal to $p + r$,
67. having a direction parallel to that of the components, and passing through a point D, such that $p \times AD = r \times CD$.

The two forces ϱ', q, are in the same plane, the intersection of which with the plane XZ is BD. These may accordingly be reduced to a single one ϱ, equal to $\varrho' + q$, that is, equal to

$$p + q + r,$$

having a direction parallel to that of ϱ' and r, and passing through

a point E, such that $\varrho' \times DE = q \times BE$. It follows, therefore, from this and what is said above, that *any number of forces, the directions of which are parallel, may be reduced to a single one, equal to the sum of those which act in one direction, minus the sum of those which act in the contrary direction, whether the given forces are in the same or in different planes.*

Let us now inquire more particularly how we are to determine through what point the resultant passes.

If from the points A, D, C, B, E, we draw the lines AA', DD', CC', BB', EE', perpendicularly to the common intersection of the two planes XZ, ZV; on account of the parallels AA', DD', CC', we shall have

$$AD : CD :: A'D' : C'D',$$

Now the equation $p \times AD = r \times CD$, found above, gives

$$AD : CD :: r : p;$$

hence

$$A'D' : C'D' :: r : p,$$

and consequently

$$p \times A'D' = r \times C'D'.$$

In like manner from the parallels DD', EE', BB', we obtain,

$$DE : BE :: D'E' : B'E';$$

and from the equation $\varrho' \times DE = q \times BE$, we have the proportion

$$DE : BE :: q : \varrho';$$

therefore

$$\begin{aligned} D'E' : B'E' &:: q : \varrho' \\ &:: q : p + r; \end{aligned}$$

whence

$$(p + r) \times D'E' = q \times B'E'.$$

Let us now take in the intersection ZF of the two planes, a fixed point F, and seek the distance FE' of this point from the point E', corresponding to E, through which the resultant passes. It is clear that

$$A'D' = FD' - FA', \quad C'D' = FC' - FD',$$
$$D'E' = FE' - FD', \quad B'E' = FB' - FE'.$$

Substituting these values for their equals in the two equations,

$$p \times A'D' = r \times C'D', \quad (p+r) \times D'E' = q \times B'E',$$

we shall have

$$p \times FD' - p \times FA' = r \times FC' - r \times FD',$$

and

$$(p+r) \times FE' - (p+r)\, FD' = q \times FB' - q \times FE'.$$

The first of these two equations gives

$$(p+r) \times FD' = p \times FA' + r \times FC';$$

substituting this value for its equal in the second equation, we obtain

$$(p+r) \times FE' - p \times FA' - r \times FC' = q \times FB' - q \times FE';$$

or, the terms multiplied by FE' being collected into one factor,

$$(p+q+r) \times FE' = p \times FA' + q \times FB' + r \times FC',$$

which gives

$$FE' = \frac{p \times FA' + q \times FB' + r \times FC'}{p+q+r}.$$

Now this expression for the distance FE' is precisely that which we should have found for the distance at which the resultant passes, if the three forces p, q, r, had all been in the plane ZV, and had passed through the points A', C', B', corresponding to the points A, C, B, through which they actually pass. If, therefore, we imagine the straight line FX perpendicular to the plane ZV, we shall find the distance FE', of the resultant from this straight line, by taking the sum of the moments with reference to this line (as if the forces, retaining their distances respectively from this line, were all in the plane ZV, to which this line is perpendicular), and dividing this sum of the moments* by the sum of the forces.

To determine the point E, therefore, it only remains to find the distance $E'E$, or (by taking EE'' parallel to ZF) the distance FE'', at which this same force passes, from ZF. Now it is manifest from what we have said with respect to the distance FE', that in

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* It must be observed, once for all, that by the general term, *sum of the moments*, is to be understood the sum of the moments of the forces that tend to turn the system in one direction, minus the sum of those which tend to turn it in the opposite direction. By *sum of the forces*, also, is to be understood the sum of those which act in one direction, minus the sum of those which act in the opposite direction.

order to find the distance FE'', we have only to imagine a plane passing through ZF, perpendicular to the direction of the forces, and then to take the sum of the moments with respect to ZF (the intersection of this plane with the plane ZV), as if the forces, without changing their distances respectively from the plane ZV, were all in the plane XV, and to divide this sum of the moments by the sum of the forces. We should then have every thing which is necessary for fixing the point E, through which the resultant passes. **Top. 1.**

Of Forces the Directions of which are neither in the same Plane nor parallel to each other.

71. Let p, q, r, be three forces directed in the manner repre- **Fig. 24.**
sented in the figure, and situated in three different planes. Suppose any plane XZ meeting in H the direction of p, in F the direction of q, and in L the direction of r. As a force may be considered as applied at any point whatever in its direction, let us suppose the three forces to be applied at the points H, F, L, and to be represented by the lines HV, FT, LK, prolongations respectively of the directions of the several forces below the plane XZ. Let us also imagine planes passing through the lines AH, BF, CL, perpendicularly to the plane XZ, the intersections with XZ being represented by the straight lines GHN, EFY, DLM. This premised, it is evident that we may decompose each of the forces in question into two others, one of which shall be in the plane XZ, and the other perpendicular to this plane. We can, for example, decompose the force HV into two others, one directed according to HN, and the other according to HO, perpendicular to the plane XZ; so that for the three forces HV, FT, LK, may be substituted the six forces

$$HN,\ FY,\ LM,\ HO,\ FS,\ LI,$$

the three first of which are in the plane XZ, and the three last perpendicular to this plane.

Now the three forces HN, FY, LM, may be reduced to a
single one, which shall also be in the plane XZ; and the three HO, 69.
FS, LI, may likewise be reduced to a single one, which shall be
perpendicular to the plane XZ. Accordingly, *whatever be the num-* 70.
ber of given forces, and whatever their directions, we may always

reduce them to two at the most, one being in the direction of a known plane, and the other perpendicular to this plane.

Although the demonstration of this proposition may appear to be adapted only to those cases where all the forces meet the plane XZ, it will be seen, with a little attention, that it has a general application. For, after having reduced all the forces that meet this assumed plane to two, we may conceive this plane, without ceasing to meet these two resultants, so placed as to coincide with the directions of those that were at first parallel to it; and the given forces must be either parallel to the assumed plane, or such as being produced will meet it.

72. With respect, therefore, to forces exerted in different planes, the result is not the same as that with respect to forces whose directions are in the same plane. The latter forces, as we
70. have seen, may always be reduced to a single one. The former are reduced to two, which do not admit of being represented by one, except in the case where the resultant of the forces which act in the plane XZ, happens to meet the resultant of the forces perpendicular to this plane.

73. We can accordingly, in the manner above explained, find the two resultants of any number of forces directed in different planes. But, although the method here pursued may be useful in certain cases, there are many in which it is not the most convenient. We proceed, therefore, to make known another.

Fig. 25. Let p be any one of the proposed forces, and AB the line which represents it. From any fixed point X draw the three straight lines XZ, XY, XT, perpendicular each to the plane of the other two. These mutually perpendicular lines are called *rectangular co-ordinates.* If now, upon AB as a diagonal, we form the rectangular parallelogram $ADBC$, having its plane perpendicular to the plane YXT, and its side BC parallel to XZ; and then upon BD as a diagonal we form the rectangular parallelogram $DFBE$, having its plane parallel to the plane YXT, and its sides BF, BE, parallel to the straight lines XT, XY, respectively; it is evident 1. That for the force AB we may substitute the force BC parallel to XZ, or perpendicular to the plane YXT, and the force BD parallel to this latter plane; 2. That for the force BD we may substitute the force BE parallel to XY, or perpendicular to the plane ZXT, and

the force BF parallel to XT, or perpendicular to the plane ZXY; so that the force p, or AB, is decomposed into three forces parallel to three rectangular co-ordinates, or (which is the same thing) into three forces perpendicular to three mutually perpendicular planes.

Now what has been said of the force p is evidently applicable
to any other force not perpendicular to one of the three planes.
If therefore all the forces like p, be considered as thus decom-
posed; and we afterwards reduce to a single force all the forces
perpendicular to the plane ZXT, the same thing being done with 69.
respect to all the forces perpendicular to the plane ZXY, and also
with respect to all the forces perpendicular to YXT, it will be seen
that we may reduce any number of forces, directed in different
planes, to three forces perpendicular to three planes, these planes
being perpendicular respectively to each other.

If either of the given forces happen to be parallel to one of the straight lines XZ, XY, XT, its components parallel to the two other straight lines, as obtained by the above method, would be each equal to zero.

Such are the general principles of the composition and decomposition of forces.

IX

Of the Centre of Gravity.

74. Before we proceed to treat of the particular effects produced by the forces whose general properties we have been considering, it is necessary to speak of the centre of gravity, a subject of the greatest importance in all inquiries relating to the motion of machines and bodies of a known structure.

By gravity, we mean the force which urges bodies downward in vertical lines, or directions perpendicular to the surface of tranquil waters. If the earth, or the surface of these waters, were perfectly spherical, the directions of gravity would all meet at the centre. But although this surface is not perfectly spherical, the deviation is so inconsiderable, that, for the objects we have more immediately in view, we may, without sensible error, regard the directions of gravity as meeting at the centre of the earth.

The earth being considered as a sphere, its radius is estimated at 3956 miles, or 20887680 feet. From this it may be read-

ily inferred, that it would require at the surface of the earth an extent of about 100 feet to subtend an angle of one second at the centre. Thus, with respect to a machine of 100 feet in length, the directions of gravity at the extremities would want only one second of being parallel. Hence, on account of the great distance of the centre of the earth, compared with the dimensions of any machine, *we may regard the directions of gravity as parallel.* We may also for the same reason, *consider the force of gravity, exerted upon different parts of the same body, as the same in point of magnitude, and capable of giving the same velocity in the same time.*

75. By the *centre of gravity* of a body, or *system* of bodies (that is, any assemblage whatever of bodies), we mean that point through which passes the resultant of all the particular forces exerted by the gravity of the several parts of the body, or system of bodies, in whatever position the body or system is placed.

Fig. 26. If, for example, in the actual position of the triangle ABC, the resultant force of all the actions of gravity, upon the several parts of this triangle, passed through a certain point G of its surface, and in another position $a\ b\ c$, it should pass through the same point G, this point is what we call the centre of gravity. We shall see hereafter that the resultant in question passes through the same point in all possible positions of the given body.

76. The centre of gravity is easily determined by means of what we have said upon the use of moments in finding the resul-
61. tant of several parallel forces.

Fig. 27. Let there be any number of bodies m, n, o, whose masses we will consider for the present as concentrated in the points A, B, C, situated in the same plane. Let u be the velocity which gravity tends to give to each in an instant;

$$u \times m,\ u \times n,\ u \times o, \quad \text{or} \quad u\,m,\ u\,n,\ u\,o,$$

will be the quantities of motion, or forces, with which the bodies tend to move according to the parallel directions $A''A$, $B''B$, $C''C$. In order, therefore, to find the position of the resultant, we take the sum of the moments with regard to any point F, assumed at pleasure, in a line perpendicular to the directions of the forces, and divide this sum by the sum of the forces; we have therefore for the
66. value of the distance FG'', at which this resultant passes,

$$FG'' = \frac{u\,m \times FA'' + u\,n \times FB'' + u\,o \times FC''}{u\,m + u\,n + u\,o},$$

or, by suppressing the common factor u,

$$FG'' = \frac{m \times FA'' + n \times FB'' + o \times FC''}{m + n + o}.$$

In like manner, if we draw the lines AA', BB', CC', parallel to FG'', and terminating in the vertical FC'; and suppose moreover, that the point G, taken in the direction of the resultant, is the centre of gravity sought, by drawing $G'G$ also parallel to FG'', we shall have

$$FG'' = G'G, \quad FB'' = B'B, \quad FA'' = A'A, \quad FC'' = C'C;$$

whence

$$G'G = \frac{m \times A'A + n \times B'B + o \times C'C}{m + n + o};$$

that is (by considering the masses m, n, o, as representing the forces, the velocity u being common), *the distance of the centre of gravity of several bodies from an assumed straight line, is found by dividing the sum of the moments of these bodies (taken with respect to this line) by the sum of the masses.*

Let us now conceive the system of bodies m, n, o, reversed in such a manner that FA'', instead of being horizontal, shall become vertical, &c.; it is apparent, that in order to find the distance of the resultant from the line FA'', now vertical, it will be necessary to take the sum of the moments with respect to FA'', and to divide this sum by the sum of the masses; which gives

$$G''G = \frac{m \times A''A + n \times B''B + o \times C''C}{m + n + o}.$$

Having found the distance of the point G from two fixed known lines FA'', FC', the position of this centre G is evidently de- Top. 1.
termined.

It is here taken for granted that the distances $A'A$, $A''A$, $B'B$, BB'', &c., are known, since the point through which FA'', FC', are drawn, is assumed at pleasure.

77. If the distances $A''A$, $B''B$, &c., are each zero; that is, if all the bodies are in the same straight line FA'', the sum of the moments with respect to this line is zero; the distance $G''G$ is therefore zero. Accordingly, *if several bodies, considered as points, are in the same straight line, their common centre of gravity is also in this line.*

78. If the lines FA'', FC', are either of them drawn in such a manner as to have bodies situated on each side of it instead of the sum of the moments, we should say the sum of the moments that are found on one side, minus the sum of the moments that are found on the other side. As to the denominator of the fraction which expresses the distance of the centre of gravity, it will always be composed of the sum of the masses, since all the forces, by the nature of gravity, act in the same direction. What is here said is applicable to any number of bodies, which, being considered as points, are situated in the same plane.

The lines FA'', FC', are called *the axes of the moments.*

79. If now we suppose the point F, which we at first took arbitrarily, to be in G, $G'G$ and $G''G$ become each equal to zero. Therefore the sum of the moments with respect to FA'', and the sum of the moments with respect to FC', must in this case be each equal to zero.

Fig. 28. 80. We now say, that if the sum of the moments of several bodies with respect to the straight line TS, passing through the point G, is equal to zero; and the sum of the moments with respect to the straight line DE, perpendicular to TS, and passing also through G, is in like manner equal to zero; the sum of the moments with respect to any other straight line LH, passing through the same point G, will also be equal to zero.

Indeed, having let fall upon the lines DE, TS, LH, the perpendiculars AA', AA'', AA'''; if we suppose that the point I is that in which AA' meets LH, from the right-angled triangle $GA'I$, we have

$$\sin GIA' : GA' :: \sin DGL : A'I,$$

or

$$\cos DGL : AA'' :: \sin DGL : A'I = \frac{AA'' \sin DGL}{\cos DGL};$$

whence

$$AI = AA' - A'I = AA' - \frac{AA'' \sin DGL}{\cos DGL}.$$

Now from the right-angled triangle IAA''', we have, radius being supposed equal to 1,

$$1 : AI :: \sin AIA''' : AA'''$$
$$:: \cos DGL : AA''' = AI \times \cos DGL;$$

that is, substituting for AI its value above found,

$$AA''' = AA' \cos DGL - AA'' \times \sin DGL;$$

hence, if we multiply by the mass m to obtain the moment, we shall have

$$m \times AA''' = m \times AA' \times \cos DGL - m \times AA'' \times \sin DGL;$$

in other words, the moment of the body m with respect to the axis LH, is equal to the cosine of the angle DGL, multiplied by the sum of the moments with respect to the axis DE, minus the sine of the same angle DGL, multiplied by the sum of the moments with respect to the axis TS.

Now it is manifest, that with regard to any other body n, we should arrive at a similar result, with the exception only of the signs according to which the bodies are on the same or on different sides of LH. Consequently, if we take the sum of all the moments with respect to the axis LH, we shall find that it is equal to the cosine of the angle DGL, multiplied by the sum of the moments with respect to DE, minus the sine of the angle DGL, multiplied by the sum of the moments with respect to TS. But each of these two last sums is by supposition equal to zero; consequently their products by the cosine and sine respectively of the angle DGL, will be each equal to zero; therefore, also, *the sum of the moments with respect to any axis whatever* LH, *which passes through the centre of gravity* G, *is equal to zero.*

81. Hence we infer that the resultant action of all the particular actions of gravity, which are exerted upon the several parts of a system of bodies, passes always through the same point of this system, whatever be its position; for it is not with respect to the direction of the resultant that the sum of the moments of the several parallel forces may be equal to zero. 68.

Moreover, although the inquiry hitherto has been only respecting bodies whose centres of gravity are in the same plane, the method is not the less applicable to the case where the parts of the system are in different planes.

82. If the bodies, still regarded as points, are not in the same plane, let us imagine a horizontal plane XZ, and from each of the Fig. 23. gravitating points p, q, r, let the vertical lines $A\,p$, $B\,q$, $C\,r$, be supposed to be drawn; and in order to determine the point E,

through which passes the resultant $\varrho\, E$, in the direction of which must be the centre of gravity, we take the moments with respect to two fixed lines FX, FZ, assumed in the horizontal plane, perpendicular to each other; we take, I say, the sum of the moments, as if the bodies were all in this horizontal plane; and having divided each of the two sums of moments by the sum of the masses or forces p, q, r, we shall have the two distances $E'E$, $E''E$. It will only remain, therefore, to find at what distance EG, below the horizontal plane this centre is situated.

Now if we imagine the figure reversed, the plane XZ becoming vertical, and ZV horizontal, it will be seen that in order to determine the distance $E'G'$, corresponding and equal to EG, the distance sought, it is necessary, according to the method above pursued, to take the sum of the moments with respect to ZF, as if the bodies were all in the plane ZV, and to divide this sum by the sum of the masses; we have then every thing that is requisite in order to fix the position of the centre of gravity.

83. Hence, by recapitulating what we have said, this problem reduces itself to the following particulars;

(1.) When the several bodies, considered as points, are situated
Fig. 29. in the same straight line, we take the sum of the moments with respect to a fixed point F, assumed arbitrarily in this line, and divide this sum by the sum of the masses, and the quotient will be the distance of the centre of gravity G from the point F.

(2.) When the several bodies, considered as points, are all in the
Fig. 27. same plane; through a point F, taken arbitrarily in this plane, we suppose two lines FA'', FC', to be drawn at right angles to each other; and having let fall perpendiculars upon each of these two lines from each gravitating point, we imagine that these gravitating points are applied successively to the lines FA'', FC', where their perpendiculars respectively fall. We then seek, as in the case just stated, what would be the centre of gravity G'' in FA'', and what would be the centre of gravity G' in FC'; drawing lastly through these two points the lines $G''G$, $G'G$, parallel respectively to FC', FA'', and their point of meeting G will be the centre of gravity sought.

(3.) When the several bodies, considered as points, are in
Fig. 23. different planes, we imagine three planes, one horizontal, and the

two others vertical and perpendicular to each other. From each gravitating point we suppose perpendiculars let fall upon each of these three planes; we then take the sum of the moments with respect to each plane, and dividing each of these sums by the sum of the masses, we shall have the three distances of the centre of gravity from the three planes respectively.

84. It must be recollected, moreover, in what is above said, that when the bodies are on different sides of the line or plane with respect to which the moments are considered, it is necessary to take with contrary signs the moments of bodies that are found on opposite sides.

85. We will here make a remark, that is suggested by what has been said, and which will enable us to abridge, in many cases, the process of finding the centre of gravity as well as the solution of other problems.

Since the distance of the centre of gravity is expressed by the sum of the moments divided by the sum of the masses, if this centre happen to be in the point, line, or plane, with respect to which the moments are considered, the distance being zero, the sum of the moments must also be zero. Therefore, *the sum of the moments with respect to any such plane as may pass through the centre of gravity is zero.*

86. Hitherto we have considered bodies as so many points, and we have seen how the centre of gravity of all these points may be determined, whatever be their number and position. Now a body of any size or figure whatever, being only an assemblage of other bodies or material parts, which may be considered as points, it follows that, by the method above pursued, we may determine the centre of gravity of a body of any figure whatever.

Also, since the centre of gravity is simply the point through which passes the resultant of all the particular efforts made by the several parts of a body in virtue of their gravity, and since this resultant is equal to the sum of all these particular efforts; it follows, that we may in all cases suppose the whole weight of a body united at its centre of gravity, and the weight would have the same effect upon this point, when thus united, that it would have in its actual state of distribution through all parts of the body.

87. When, therefore, it is proposed to find the common centre of gravity of several masses of whatever figure, we begin by seeking the centre of gravity of each of these masses, which is attended with no difficulty. Then, the weight of these masses being considered as united each at its centre of gravity, we seek the common centre of gravity, as if all these bodies were points situated where each has its particular centre of gravity.

88. Accordingly, every thing which we have said hitherto upon the common centre of gravity of several bodies, considered as points, is equally applicable to bodies of whatever figure, if we take, in estimating the moments, instead of the distance of each body, the distance of its particular centre of gravity.

89. Hence, finally, *if several bodies, of whatever figure, have their particular centres of gravity in the same straight line, or in the same plane; their common centre of gravity will, in the former case, be in the given straight line, and in the latter in the given plane.*

Application of the Principles of the Centre of Gravity to particular Problems.

Fig. 30. 90. Let AB be a straight line uniformly heavy. It will be seen at once without the aid of any demonstration, that the middle point P, of its length will be its centre of gravity. But in order to illustrate and confirm the theory of moments, developed in the preceding articles, let us seek the centre of gravity according to the principles of this method.

We imagine this straight line divided into an infinite number of points, of which Pp represents one; and that each is multiplied by its distance from a fixed point, as the extremity A for example. We then take the sum of these products, and divide it by the sum of the parts of which Pp is one, that is, by the line AB. Accordingly, if we call AB, a; AP, x; we shall have $Pp = dx$;
Cal. 7. and the moment of Pp will be equal to $x\,dx$, which must be integrated to obtain the sum of the moments. This sum therefore
Cal. 82. will be equal to $\frac{x^2}{2}$; and in order to have it for the whole extent of the line, we must suppose $x = a$, which gives $\frac{a^2}{2}$ for the entire sum

of the moments. Dividing this by the sum a of the masses, we have $\frac{a^2}{2a}$ or $\frac{a}{2}$ for the distance of the centre of gravity from the point 83.
A. Thus, *the centre of gravity of a straight line, uniformly heavy, is its middle point*, as was before manifest.

91. Hence, (1.) *In order to have the centre of gravity of the* Fig. 31.
perimeter of a polygon, it is necessary, from the middle of each of the sides, to let fall perpendiculars upon two fixed lines AB, AC, taken in the plane of this polygon; and, considering the weight of each side as united in the middle of this side, to seek the 86.
common centre of gravity of these weights in the manner already explained.

92. (2.) *The centre of gravity of the surface of a parallelogram is the middle point of the line which joins the middle of two opposite sides.* For, by considering the parallelogram as composed of Fig. 32.
material lines, parallel to these two sides, each will have its centre of gravity in the line which passes through the middle of these same sides. The common centre of gravity, therefore, of all the lines will be in the bisecting line. It will, moreover, be in its middle point, since this line, considered as sustaining the weights of all the 90.
other lines, is uniformly heavy.

93. (3.) *To find the centre of gravity of a triangle ABC*; we Fig. 33.
draw from the vertex A to the middle D of the opposite side BC, the straight line DA, and from the point D we take $DG = \frac{1}{3} DA$.

Indeed, the straight line DA, which divides BC into two equal parts at the point D, divides also into two equal parts every other line LN, parallel to BC; accordingly, if we consider the surface of the triangle as an assemblage of material lines parallel to BC, the line DA, which passes through the particular centres of gravity of all these lines, will also pass through their common centre of gravity, that is, through the centre of gravity of the triangle. For 77.
the same reason, the line CE, which passes through the middle of AB, will in like manner pass through the centre of gravity of the triangle. This centre is consequently at the point of intersection G of the two lines CE and DA. Now, if we join ED, it will be parallel to AC, since it divides into two equal parts the sides AB,

Geom. 199. Geom. 202. BC. The two triangles EGD, AGC, are accordingly similar, as well as the triangles ABC, EBD; we have, therefore,

$$DG : AG :: DE : AC :: BD : BC :: 1 : 2;$$

that is, DG is half of AG, and therefore one third of AD.

94. Hence, *in order to find the centre of gravity* G *of a trape-*
Fig. 34. *zoid*, we draw KL through the middle points of the two parallel sides, and from these same points K, L, we draw the lines KA, LD, to the vertices of the opposite angles A, D; then having taken

$$KE = \tfrac{1}{3} KA, \quad LF = \tfrac{1}{3} LD,$$

we join EF, which will cut KL in G, the point sought.

For, by reasoning as we have done in the case of the triangle, we shall see that the centre of gravity G must be in KL. Moreover, since E, F, are the centres of gravity respectively, of the
93. triangles CAD, ADB, which compose the trapezoid $ABDC$, the common centre of gravity of the two triangles or of the trapezoid
77. must be in EF; it follows, therefore, that it is at the intersection G.

To find the distance LG, which we shall have occasion to use hereafter, we draw the lines EH, FI, parallel respectively to AB; and since

$$KE = \tfrac{1}{3} KA, \quad \text{and} \quad LF = \tfrac{1}{3} LD,$$

we shall have

$$EH = \tfrac{1}{3} AL, \quad \text{and} \quad FI = \tfrac{1}{3} KD,$$

or

$$EH = \tfrac{1}{6} AB, \quad \text{and} \quad FI = \tfrac{1}{6} CD.$$

For the same reason,

$$KH = \tfrac{1}{3} KL, \quad LI = \tfrac{1}{3} KL;$$

therefore

$$HI = \tfrac{1}{3} KL.$$

Now the similar triangles GHE, GFI, give

$$EH : GH :: FI : GI;$$

whence

$$EH + FI : GH + GI \quad \text{or} \quad HI :: FI : GI;$$

that is,

$$\tfrac{1}{6} AB + \tfrac{1}{6} CD : \tfrac{1}{3} KL :: \tfrac{1}{6} CD : GI;$$

therefore

$$GI = \frac{\frac{1}{3} KL \times \frac{1}{6} CD}{\frac{1}{6} AB + \frac{1}{6} CD} = \frac{\frac{1}{3} KL \times CD}{AB + CD};$$

and, because $LG = LI + GI$, if we substitute for LI and GI the values above found, we shall have

$$LG = \tfrac{1}{3} KL + \frac{\frac{1}{3} KL \times CD}{AB + CD}$$
$$= \frac{\frac{1}{3} KL \times (AB + 2\ CD)}{AB + CD}.$$

We remark in passing, that if the height KL of the trapezoid were infinitely small, and the difference of the two sides AB, CD, were infinitely small, these sides must be considered as equal, so that the distance LG would reduce itself to $\frac{\frac{1}{3} KL \times 3\ AB}{2\ AB}$ or $\frac{1}{2} KL$; that is, the centre of gravity in this case is equally distant from the two opposite bases.

95. *To find the centre of gravity of the surface of any polygon,* Fig. 35.
we divide it into triangles, and having found the centre of gravity
of each triangle in the manner above shown, we determine the com- 93.
mon centre of gravity of all the triangles, by considering them as
so many masses proportional to their surfaces, and concentrated each
at its particular centre of gravity, agreeably to the method already 86.
adopted.

It will hence be seen how we should proceed in determining the centre of gravity of the surface of any solid figure terminated by plane surfaces.

96. In fine, it is not always necessary to have recourse to moments in finding the centre of gravity. If it were proposed, for example, to determine the centre of gravity of the perimeter of a
regular pentagon $ABCDE$, I should draw from the vertex of one Fig. 36.
of its angles A, to the middle of its opposite side, a straight line AH; likewise from the vertex of another angle E, to the middle of its opposite side, a straight line EI, and the intersection G of these lines will be the centre of gravity.

Indeed the common centre of gravity of the two sides AB, AE, is evidently the middle c, of the line $b\ a$, which passes through their middle points. The common centre of gravity of the two sides

BC, DE, is for the same reason the middle of the line IK which passes through their middle points; and the side CD has its centre of gravity in H. Now it is manifest that the line AH passes through the middle points c, e, and H; it accordingly passes through the common centre of gravity of the five sides. It may be shown, in like manner, that IE also passes through the centre of gravity; therefore this centre is at the intersection G of AH and IE.

97. By pursuing the same kind of reasoning which we adopted in the case of the triangle, it might be demonstrated that the point G is the centre of gravity of the surface of a regular pentagon.

In general, it may be shown, by the same method, that the centre of gravity of the perimeter, as well as of the surface of any regular polygon, of an odd number of sides, is the point of intersection of two straight lines, each of which is drawn from the vertex of one of the angles to the middle of the opposite side.
Fig. 37. And when the number of sides is even, the centre of gravity, both of the perimeter and of the surface, is the point of intersection of two straight lines drawn through the middle points of two pairs of opposite sides. We might also extend this mode of reasoning to the circle, by regarding it as a polygon of an infinite number of sides, and we should find that the centre of gravity of the circumference, and of the surface, is the centre.

When the number of lines, surfaces, bodies, &c., is not considerable, the centre of gravity may be found by the method of
Fig. 38. articles 53, 54. Let the three points A, B, C, for example, be the centres of gravity of three lines, or three surfaces, or three bodies, whose weights are represented by the masses m, n, o. Having joined two of these points, as B and C, by the line BC, we divide BC at G', in such a manner as to give

$$n : o :: CG' : BG'$$

or

$$n + o : n :: CB : CG';$$

and the point G' thus found, will be the centre of gravity of the two weights n, o. We now draw $G'A$, and supposing the two masses n, o, united in G', we divide, in the same way, $G'A$ in the inverse ratio of the two masses m and $n + o$, that is, so as to give

$$n + o : m :: AG : G'G,$$

or

$$n + o + m : m :: AG' : G'G;$$

and the point G will be the common centre of gravity of the three weights m, n, o. We might proceed in a similar manner with a greater number of bodies.

98. It would be easy to deduce from what precedes an easy method of finding the centre of gravity of the surface and of the solidity of any cylinder or prism. Indeed it is evident that this centre must be the middle of the line that passes through the centres of gravity of the two opposite bases, since bodies of this form, being composed of laminæ or material planes, perfectly equal and similar to the base, may be considered as so many equal weights uniformly distributed upon this line.

99. *To find the centre of gravity* G *of a triangular pyramid* SABC, *we draw from the vertex to the centre of gravity* F *of the* Fig. 39.
base, the straight line SF, *and take in this line, reckoning from* F, *the part* FG $= \frac{1}{4}$ FS.

To show that G is the centre of gravity required, from the middle D of the side AB, we draw DC, DS, to the opposite vertices C, S, of the pyramid, and having taken $DF = \frac{1}{3} DC$, and $DE = \frac{1}{3} DS$, the points F, E, are respectively the centres of gravity of the two triangles ABC, ASB. 93.

This being supposed, if we consider the pyramid as composed of material planes, parallel to ABC, the line SF, which passes through the point F, of the base, will pass through a point similarly placed in each of the parallel planes or strata. Thus the particular Geom.
centres of gravity of the several parallel planes will all be in the 408.
line SF. For the same reason the particular centres of gravity of the several planes parallel to ABS, of which we may suppose the pyramid in like manner composed, are all in the line EC. Accordingly, the centre of gravity of the pyramid is the point G, where the two lines SF, EC, situated in the plane SDC, intersect each other. Now if we draw FE, it will be parallel to CS, since DF being a third of DC, and DE a third of DS, these two lines are cut proportionally. The two triangles FEG, GCS, are there- Geom.
fore similar; and the two triangles DFE, DCS, are also similar; 199.
whence

$$FG : GS :: FE : CS :: DF : DC :: 1 : 3;$$

that is, FG is a third of GS, and consequently a fourth of FS.

100. As any solid may be decomposed into triangular pyramids, knowing the centre of gravity of a triangular pyramid, it will be easy, by the method of moments, to find the centre of gravity of any body whatever.

101. Such is the general manner of finding the centre of gravity of bodies, the parts of which are independent of each other, or rather when we have not the expression of the law by which they are connected together.

But when the parts of a figure or body have a relation that can be expressed by an equation, the centre of gravity may be found much more readily.

Fig. 40. 102. Let it be required to find the centre of gravity G, of any arc of a curve AM; we imagine an infinitely small arc $M\ m$, and take for the axis of the moments any line CN, parallel to the ordinates, which are supposed to be perpendicular among themselves. Suppose, moreover, that the distance of C from the origin A of the abscissas $= h$, h being taken of any magnitude at pleasure. To obtain the distance GG' of the centre of gravity from the axis CN, we
76. must take the sum of the moments of the arcs $M\,m$, and divide it by the sum of the arcs $M\,m$; that is, by the arc AM. Now the arc $M\,m$ being infinitely small, the distance of its middle point n, from the straight line CN, may be considered as equal to MN. We shall have, therefore,

$$Mm \times MN$$

for the moment of this infinitely small arc. But, calling AP, x,
Cal. 97. and PM, y, we shall have $M\,m = \sqrt{d\,x^2 + d\,y^2}$, and

$$MN = PC = h - x;$$

therefore $(h - x)\sqrt{d\,x^2 + d\,y^2}$ is the moment of the arc $M\,m$; and consequently $\int (h - x)\sqrt{d\,x^2 + d\,y^2}$, or the integral of $(h - x)\sqrt{a\,x^2 + d\,y^2}$, is the sum of the moments of all the infinitely small arcs $M\,m$, of which the arc AM is composed. We have, therefore,

$$GG' = \frac{\int (h - x) \sqrt{d x^2 + d y^2}}{AM}.$$

With respect to the arc AM, which is a divisor in this quantity, we have given a method of determining it exactly, when that can be done; and another method of determining it by approximation. Cal. 96. Cal. 110.

By a course of reasoning similar to the above, we should find that the distance GG'', of the centre of gravity from the axis AP, is $\int \frac{y \sqrt{d x^2 + d y^2}}{AM}$.

Such are the general formulas which serve to determine the centre of gravity of any arc of a curve of which we have the equation, by means of the lines designated by x and y.

103. If the arc of which we wish to find the centre of gravity, is composed of two equal and similar parts AM, AM', situated on each side respectively of the axis of the abscissas, it is evident that the centre of gravity G, will be in the straight line AP; we have, therefore, only to find its distance from the point C. Now it is plain that the moments of the two arcs Mm, $M'm'$, with respect to the axis NN', being equal, the distance CG will be equal to Fig. 41.

$$\frac{2 \int (h - x) \sqrt{d x^2 + d y^2}}{MAM'}.$$ 76.

For example, let the arc MAM' be an arc of a circle; we have $y = \sqrt{a x - x^2}$, a being the diameter. We shall easily find, and indeed we have already seen, that Trig. 101. Cal. 98.

$$\sqrt{d x^2 + d y^2} = \frac{\frac{1}{2} a\, d x}{\sqrt{a x - x^2}}.$$

We shall have, therefore,

$$2 \int (h - x) \sqrt{d x^2 + d y^2} = \frac{2 \int \frac{1}{2} a (h - x)\, d x}{\sqrt{a x - x^2}}$$

$$= a \int (h - x)\, d x\, (a x - x^2)^{-\frac{1}{2}}.$$

Supposing now, for the sake of greater simplicity, that the point C is the centre, then $AC = h = \frac{1}{2} a$; we shall have, therefore,

Cal. 88.
$$2\int (h-x)\sqrt{d x^2 + d y^2} = a \int (\tfrac{1}{2} a - x)\, d x\, (a x - x^2)^{-\frac{1}{2}}$$
$$= a\sqrt{a x - x^2};$$

an integral to which no constant is to be added, because when $x = 0$, this integral becomes zero; as indeed it ought, since the sum of the moments is then evidently nothing.

We have, therefore, finally,

$$2\int (h-x)\ \sqrt{d x^2 + d y^2} = a\sqrt{a x - x^2},$$

and consequently,

$$CG = \frac{a\sqrt{a x - x^2}}{MAM'} = \tfrac{1}{2} a \times \frac{2\sqrt{a x - x^2}}{MAM'} = \frac{CA \times MM'}{MAM'},$$

which gives this proportion,

$$MAM' : MM' :: CA : CG.$$

Thus we obtain the following rule; *that the distance of the centre of a circle from the centre of gravity of any arc of this circle, is a fourth proportional to the length of the arc, its chord, and radius.*

These formulas may be applied to any other curve.

We pass now to the consideration of the centre of gravity of plane surfaces bounded by curved lines.

Fig. 42. 104. Let it be required to find the centre of gravity of the surface APM; and let G represent this centre. In order to obtain the distance GG', it is necessary to take the sum of the moments of the small trapezoids $MP\,p\,m$, with respect to CN, and to divide this sum by the sum of the trapezoids, that is, by the surface APM. Now the centre of gravity F of this small trapezoid must be in the middle point of the straight line $n\,K$, equally distant from MP and $m\,p$, which point we can suppose to be in MP, on account of the infinitely small height $P\,p$. We shall have, therefore, $FL = CP$; and the moment of $P\,p\,m\,M$ will be

$$P\,p\,m\,M \times CP,$$

that is, $(h-x)\,y\,d\,x$, calling always CA, h, and AP, x. Therefore the sum of the moments will be $\int (h-x)\,y\,d\,x$, and consequently the distance

$$GG' = \frac{\int (h-x)\,y\,d\,x}{APM}.$$

It will be found, likewise, that the distance

$$GG'' = \frac{\int \frac{1}{2} y^2 \, d\, x}{APM}.$$ 44.

105. The centre of any plane surface may be found, in the same manner, by decomposing it into infinitely small trapezoids.

For example, let the surface in question be the triangle ANN', and take the base NN' and the height AC for the axes of the moments; now calling AP, x, MM', y, and AC, h, we shall have Fig. 43.
$MM'\,m'\,m = y\,d\,x$; and the moment of this trapezoid, with respect to NC, will be $(h - x)\, y\, d\, x$. Therefore, the distance GG' of the centre of gravity from the base, will be $\frac{\int (h-x)\, y\, d\, x}{AMM'}$. Now calling c the base, we have

$$AC : AP :: NN' : MM';$$

that is,

$$h : x :: c : y = \frac{c\,x}{h};$$

therefore $\int (h - x)\, y\, d\, x$ becomes $\int (h - x) \frac{c\,x\,d\,x}{h}$, or

$$\frac{\int c}{h} \left(h\, x\, d\, x - x^2\, d\, x\right),$$

the value of which is $\frac{c}{h} \left(\frac{h\,x^2}{2} - \frac{x^3}{3}\right)$ or $\frac{c\,x^2}{6\,h} (3\,h - 2\,x)$. Now the surface AMM' is $\frac{MM' \times AP}{2}$ or $\frac{c\,x^2}{2\,h}$; therefore the distance GG' of the centre of gravity of the surface from the base is

$$\frac{\frac{c\,x^2}{6\,h}(3\,h - 2\,x)}{\frac{c\,x^2}{2\,h}} \text{ or } \tfrac{1}{3}(3\,h - 2\,x),$$

which, when $x = h$, becomes $\frac{1}{3}\,h$, to which this distance is therefore equal. Now if we draw the line AGL, the similar triangles ACL, $GG'L$, give

$$LG : LA :: GG' : AC :: \tfrac{1}{3}\,h : h :: 1 : 3$$

therefore $LG = \frac{1}{3}\,LA$, which agrees with what has been before 93.
demonstrated.

106. Let us now apply the formulas to curved lines. Suppose that APM is a portion of a circle whose diameter is a, and whose
Fig. 44. centre is C; we have then $h = AC = \frac{1}{2}a$. Now

$$y = \sqrt{ax - x^2};$$

the quantity $\int (h - x)\, y\, dx$, becomes therefore

Cal. 88.

$$\int (\tfrac{1}{2}a - x)\, dx \sqrt{ax - x^2},$$

or $\int (\frac{1}{2}a - x)\, dx\, (ax - x^2)^{\frac{1}{2}}$, which is an integrable quantity, and being integrated, gives $\frac{1}{3}(ax - x^2)^{\frac{3}{2}}$; a quantity to which no constant is to be added, because it becomes zero, when $x = 0$; as it evidently ought. We have, therefore,

$$GG' = \frac{\frac{1}{3}(ax - x^2)^{\frac{3}{2}}}{APM} = \frac{\frac{1}{3}\overline{PM}^3}{APM}.$$

With respect to GG'', since $y = \sqrt{ax - x^2}$, we shall have,

104.

$$GG'' = \frac{\int \frac{1}{2}(ax - x^2)\, dx}{APM};$$

but $\int \frac{1}{2}(ax - x^2)\, dx$, or $\int \frac{1}{2}(ax\, dx - x^2\, dx)$,

is $\frac{1}{2}\left(\frac{ax^2}{2} - \frac{x^3}{3}\right)$ or $\frac{1}{12}x^2(3a - 2x)$;

we have, therefore,

$$GG'' = \frac{\frac{1}{12}x^2(3a - 2x)}{APM}.$$

If the question related to the entire segment, since it is evident that
Fig. 45. the centre of gravity E, must be in the radius CA, which bisects the arc, and that it must be at the same distance from NN' as the centre of gravity of each of the two semi-segments APM, APM', we shall have

$$CE = \frac{\frac{1}{3}\overline{PM}^3}{APM} = \frac{\frac{1}{24} \times 8 \times \overline{PM}^3}{APM} = \frac{\frac{1}{24}\overline{MM'}^3}{APM}$$

$$= \frac{\frac{1}{12}\overline{MM'}^3}{2\,APM} = \frac{\frac{1}{12}\overline{MM'}^3}{AMM'};$$

that is, *the distance of the centre of a circle from the centre of gravity of any one of its segments, is equal to the twelfth part of the cube of the chord, divided by the surface of this segment.*

Fig. 45. 107. The centre of gravity of a sector $CMAM'$ may be easily found, by observing that E, the centre of gravity of the segment

MAM', G that of the sector, and F that of the triangle, are all in the radius CA; and that, according to the principle of moments, the moment of the sector must be equal to the moment of the segment plus that of the triangle. We have then

$$CMAM' \times CG = MAM' \times CE + CMM' \times CF.$$

Now we have just found $CE = \frac{\frac{1}{3}\overline{PM}^3}{APM'}$, which may be changed into $\frac{\frac{2}{3}\overline{PM}^3}{2\,APM} = \frac{\frac{2}{3}\overline{PM}^3}{MAM'}$; therefore $CE \times MAM' = \frac{2}{3}\,\overline{PM}^3$. We know, moreover, that $CMM' = PM \times CP$, and that

$$CF = \tfrac{2}{3}\,CP,$$ 93.

so that $CMM' \times CF$ is reduced to $\frac{2}{3}\,PM \times \overline{CP}^2$. Substituting, therefore, these values, we have

$$\begin{aligned} CMAM' \times CG &= \tfrac{2}{3}\,\overline{PM}^3 + \tfrac{2}{3}\,PM \times \overline{CP}^2 \\ &= \tfrac{2}{3}\,PM\,(\overline{PM}^2 + \overline{CP}^2) \end{aligned}$$

$= \frac{2}{3}\,PM \times \overline{CM}^2$, on account of the right-angled triangle CPM; consequently,

$$CG = \frac{\frac{2}{3}\,PM \times \overline{CM}^2}{CMAM'}.$$

But the surface of the sector $CMAM'$, is equal to the arc MAM' multiplied by $\frac{CM}{2}$, therefore Geom. 290.

$$CG = \frac{\frac{2}{3}\,PM \times \overline{CM}^2}{\frac{MAM' \times CM}{2}} = \frac{\frac{4}{3}\,PM \times CM}{MAM'} = \frac{\frac{2}{3}\,MM' \times CA}{MAM'}.$$

That is, *the distance of the centre of a circle from the centre of gravity of any one of its sectors, is a fourth proportional to the arc, the radius, and two thirds of the chord of the arc.*

The formulas above found may be applied to any other curve, as the parabola, &c.

108. We now proceed to the consideration of curved surfaces, confining ourselves to those of solids of revolution. Reasoning, then, as in the preceding articles, it will be perceived that the centre of gravity of each elementary zone, is in the axis of revolution CA, and Fig. 46. that it must be regarded as at the centre P of one of the bases of this zone, considered as having an infinitely small breadth. But we have seen that the expression for this zone is $2\,\pi\,y\,\sqrt{d\,x^2 + d\,y^2}$, Cal. 97.

π representing the ratio of the diameter to the circumference. We shall have, therefore, (denoting always by h, the distance AC of A, the origin of the abscissas, from NN', the axis of the moments) $2\pi (h-x)\, y \sqrt{dx^2+dy^2}$ for the moment of this zone; from which it follows that the distance of G, the centre of gravity of the surface, from the point C, designating this surface by σ, will be

$$\frac{\int 2\pi (h-x)\, y \sqrt{dx^2+dy^2}}{\sigma}.$$

109. Let us suppose, in order to apply this formula, that it is proposed to find the centre of gravity of the convex surface of the
Fig. 47. right cone ANN'; we denote AP by x, PM by y, the height AC by h, CN the radius of the base by a, and the side AN by e. On account of the similar triangles, ACN, Mrm, we have

$$AC : AN :: Mr : Mm;$$

that is,

$$h : e :: dx : \sqrt{dx^2+dy^2} = \frac{e\,dx}{h}.$$

We have also, on account of the similar triangles, ACN and APM,

$$AC : CN :: AP : PM,$$

that is,

$$h : a :: x : y = \frac{ax}{h};$$

therefore

$$\int 2\pi (h-x)\, y \sqrt{dx^2+dy^2}$$

becomes

$$\int 2\pi \times (h-x) \times \frac{ax}{h} \times \frac{e\,dx}{h};$$

or

$$\int \frac{2\pi a e}{h^2} (hx\,dx - x^2\,dx);$$

of which the integral is

$$\frac{2\pi a e}{h^2}\left(\frac{hx^2}{2} - \frac{x^3}{3}\right), \text{ or } \frac{\pi a e x^2}{3h^2}(3h-2x).$$

Geom. 528. Now, the surface of the portion $AM'LMA$, or σ, is equal to $\frac{AM}{2} \times$ *circum* PM, and we have

$$AC : AP :: AN : AM, = \frac{AP \times AN}{AC};$$

therefore,

$$\sigma = \frac{AP \times AN}{2\,AC} \times \textit{circum}\ PM = \frac{x \times e}{2\,h} \times \frac{2\,\pi\,a\,x}{h} = \frac{\pi\,a\,e\,x^2}{h^2};$$

therefore the distance of the centre of gravity of the surface $AM'LMA$, from the point C is

$$\frac{\frac{\pi\,a\,e\,x^2}{3\,h^2}(3\,h - 2\,x)}{\frac{\pi\,a\,e\,x^2}{h^2}}, \text{ or } \tfrac{1}{3}(3\,h - 2\,x).$$

Therefore, when $x = h$, or $AP = AC$, we shall have the distance CG, of the centre of gravity of the whole curved surface of the cone,

$$= \tfrac{1}{3}(3\,h - 2\,h) = \tfrac{1}{3}\,h;$$

that is, the centre of gravity is found in the same manner as that of the surface of the triangle ANN'.

110. For a second example, we take the sphere. We now have Fig. 48.
$y = \sqrt{a\,x - x^2}$, a being the diameter, and

$$\sqrt{d\,x^2 + d\,y^2} = \frac{\frac{1}{2}\,a\,d\,x}{\sqrt{a\,x - x^2}}; \quad \text{Cal. 98.}$$

therefore $\int 2\,\pi\,(h - x)\,y\,\sqrt{d\,x^2 + d\,y^2}$ will become

$$\int 2\,\pi\,(h - x)\,\tfrac{1}{2}\,a\,d\,x;$$

and this expression, C being the centre, which gives $h = \frac{1}{2}\,a$, will be equal to $\int \pi\,a\,(\frac{1}{2}\,a\,d\,x - x\,d\,x)$, which, being integrated, is $\pi\,a\,(\frac{1}{2}\,a\,x - \frac{1}{2}\,x^2)$ or $\pi\,a\,x\,(\frac{1}{2}\,a - \frac{1}{2}\,x)$. Now we have found that the surface σ of the spherical segment $AMLM'A$, was $\pi\,a\,x$; Cal. 98.
we have, therefore, for the distance CG of C, the centre of the sphere, from the centre of gravity $G =$

$$\frac{\pi\,a\,x}{\pi\,a\,x}(\tfrac{1}{2}\,a - \tfrac{1}{2}\,x) = \tfrac{1}{2}\,a - \tfrac{1}{2}\,x = CA - \tfrac{1}{2}\,AP;$$

that is, the centre of gravity G is in the middle of the altitude AP of this segment. Hence we derive the general conclusion, that *the centre of gravity of the surface of a spherical zone, comprehended between two parallel planes, is the middle point of the altitude of this zone.*

111. We shall terminate this branch of our subject, with the investigation of the centres of gravity of solids.

Fig. 46. If we consider a solid as made up of laminæ infinitely thin, and parallel to each other, and represent generally by σ one of the opposite bases of each lamina, and by $d\,x$ its thickness, we shall have $\sigma\, d\, x$ as the expression for each lamina; and consequently $\sigma\,(h - x)\,d\,x$ for its moment with respect to a plane parallel to these laminæ, whose distance AC from the vertex A we represent by h. Therefore, denoting by b the bulk $ALM'MA$, the distance of the centre of gravity from C will be $= \frac{\int \sigma\,(h - x)\,d\,x}{b}$. Now the

Cal. 100, &c. value of b is determined by methods which have been heretofore given, and that of $\int \sigma\,(h - x)\,d\,x$ is found by the same methods, when the value of σ is known in terms of x. We shall thus obtain the distance of the centre of gravity from a known plane. We might find in the same way the distance of this centre from each of the two other planes, perpendicular to one another, and to the first; but we shall confine ourselves for the present to those solids, of which the parallel laminæ have their respective centres of gravity all in the same straight line, as pyramids, and solids of revolution.

112. We begin with pyramids. Let h denote the height AC
Fig. 49. of any pyramid; x the perpendicular distance AP of any lamina from the vertex; c^2 the surface of the base; we shall have the surface of the lamina situated at the distance x from the vertex, by
Geom. 409. this proportion

$$h^2 : x^2 :: c^2 : \frac{c^2\,x^2}{h^2};$$

we have, therefore,

$$\sigma = \frac{c^2\,x^2}{h^2};$$

whence $\int \sigma\,(h - x)\,d\,x$ becomes

$$\int \frac{c^2}{h^2}\,(h\,x^2\,d\,x - x^3\,d\,x) = \frac{c^2}{h^2}\left(\frac{h\,x^3}{3} - \frac{x^4}{4}\right),$$

or,

$$\frac{c^2\,x^3}{12\,h^2}\,(4\,h - 3\,x).$$

Geom. 416. But the pyramid which has x for its height, and for its base σ or $\frac{c^2\,x^2}{h^2}$ is $= \frac{c^2\,x^3}{3\,h^2}$; the distance of the centre of gravity, therefore, is

$$\frac{\frac{c^2\ x^3}{12\ h^2}(4\ h - 3\ x)}{\frac{c^2\ x^3}{3\ h^2}}$$

or $$\tfrac{1}{4}\,(4\ h - 3\ x)\,;$$

this quantity, when $x = h = AC$, is reduced to $\frac{1}{4}\ h$, and we have the height CG'' of the centre of gravity G, above the base $= \frac{1}{4}\ h$.

Now let G' represent the centre of gravity of the base; the line AG' will pass through G, the centre of gravity of the pyramid; and the parallels $G''G$, $G'C$, will give

$$G''C \text{ or } \tfrac{1}{4}\ h\ :\ AC \text{ or } h\ ::\ GG'\ :\ AG'\,;$$

whence $GG' = \frac{1}{4}\ AG'$; which confirms what we have before said, 99.
and shows that the centre of gravity of every pyramid, is one fourth of the distance from the centre of gravity of the base, to the vertex.

113. With respect to solids of revolution, the general value of Geom. 291.
σ is $\pi\ y^2$; the general expression for the distance of the centre of gravity will thus be $\frac{\int \pi\ y^2\ (h - x)\ d\ x}{b}$. This formula may be applied to the sphere, the ellipsoid, &c., and by means of the ellipsoid may be determined the centre of gravity of masts.

114. What we have said with respect to centres of gravity, will enable us to arrive at a solution in any case that may occur; we shall, notwithstanding, point out particularly the course to be pursued in order to find the centre of gravity of the immersed part of a ship's bottom, or rather of a homogeneous solid of this form.

We may suppose the centre of gravity to be in a vertical plane passing through the axis of the keel, and we have only to determine its horizontal distance from a vertical line drawn through a given point of the stern-post, and its vertical distance from the keel.

For each of these objects we must begin by determining the centre of gravity of a surface $ANDFPB$, bounded by two parallel lines AB, DF, and two equal curves, similar to AND, Fig. 50.
BPF.

If we had the equation of this curve, nothing would be more easy than to determine its centre of gravity G, by the preceding methods. But not having it, we must conceive the line CE to

pass through C and E, the middle points of AB, DF, respectively, and that this line is divided by the perpendiculars TH, KM, &c., into equal parts, so small that the arcs comprehended between any two adjacent perpendiculars, shall not differ sensibly from straight lines. We must next take the moments of the trapezoids $DTHF$, $TKMH$, &c., with respect to the point E, and
Geom. 178. divide the sum of these moments by the sum of the trapezoids, that is, by the surface $ANDFPB$. This surface, being composed of trapezoids, is readily determined. We have, therefore, only to find a simple expression for the sum of the moments. Now the distance of the centre of gravity of the trapezoid $THFD$, from the point E, is

94.

$$\frac{\frac{1}{3}\, IE \times (DF + 2\ TH)}{DF + TH};$$

that of the trapezoid $TKMH$ from the same point E, will be, for the same reason, taken in connexion with the equality of the lines IE, IL, &c.,

$$\frac{\frac{1}{3}\, IE \times (TH + 2\ KM)}{TH + KM} + IE, \text{ or } \frac{\frac{1}{3}\, IE\,(4\ TH + 5\ KM)}{TH + KM}.$$

In like manner, the distance of the centre of gravity of the trapezoid $NKMP$, will be

$$\frac{\frac{1}{3}\, IE \times (KM + 2\ NP)}{KM + NP} + 2\ IE, \text{ or } \frac{\frac{1}{3}\, IE \times (7\ KM + 8\ NP)}{KM + NP},$$

and so on.

Geom. 178. Now if we multiply each distance by the surface of the corresponding trapezoid, that is, by half the sum of the two parallel sides, multiplied by their common height IE, we shall have for the series of these moments,

$$\tfrac{1}{6}\, \overline{IE}^2 \times (DF + 2\ TH), \quad \tfrac{1}{6}\, \overline{IE}^2 \times (4\ TH + 5\ KM),$$
$$\tfrac{1}{6}\, \overline{IE}^2 \times (7\ KM + 8\ NP),$$

and so on; the sum of which will be

$$\tfrac{1}{6}\, \overline{IE}^2 \times (DF + 6\ TH + 12\ KM + 18\ NP + 24\ QS + 14\ AB).$$

It may be observed, that if there were a greater number of divisions, the multiplier of the last term, which is here 14, would be in general $2 + 3\,(n - 2)$ or $3\,n - 4$, n representing the whole number of the perpendiculars DF, TH, &c., including AB,

which may be zero. So that the general expression for the sum of the moments is reduced to

$$\overline{IE}^2\left(\tfrac{1}{6}\,DF + TH + 2\,KM + 3\,NP + 4\,QS + \&c. \ldots + \frac{(3\,n-4)}{6}\,AB\right).$$

But it is evident that the surface *ANDFPB* has for its expression,

$$IE \times \left(\tfrac{1}{2}\,DF + TH + KM + NP + \&c. \ldots + \tfrac{1}{2}\,AB\right);$$

and hence the distance of the centre of gravity G, namely,

$$EG = \frac{IE \times \left(\tfrac{1}{6}\,DF + TH + 2\,KM + 3\,NP + \&c. \ldots + \frac{(3n-4)}{6}\,AB\right)}{\tfrac{1}{6}\,DF + TH + KM + NP + \&c. \ldots + \tfrac{1}{2}\,AB}.$$

This formula, expressed in common language, furnishes the following rule;

To find the distance of the centre of gravity G, from one of the extreme ordinates DF,

(1). *Take a sixth of the first ordinate* DF; *a sixth of the last ordinate* AB, *multiplied by triple the number of ordinates less* 4; *then the second ordinate, double the third, triple the fourth, and so on; which may be called the first sum.*

(2). *To half the entire sum of the two extreme ordinates, add all the intermediate ordinates, for a second sum;*

(3). *Divide the first sum by the second, and multiply the quo-* 83.
tient by the common interval between two adjacent ordinates.

For example, if there were 7 perpendiculars, whose values were 18, 23, 28, 30, 30, 21, 0, feet; and each interval were 20 feet; I should take a sixth of 18, which is 3; and since the last term is 0, I should add to 3, the second ordinate 23, double of 28, triple of 30, 4 times 30, and so on, which would give 397. To the half of 18, I should next add, 23, 28, &c., the result of which would be 141; now dividing 397 by 141, and multiplying by 20, I should have

$$\frac{397 \times 20}{141} \text{ or } \frac{7940}{141} = 56 \text{ feet 4 inches nearly.*}$$

When we once know how to determine the centre of gravity of any section of a solid, that of the solid itself is easily found.

* See Bouguer, *Traité du Navire*, p. 213.

Hence, by means of what is above laid down, we can determine the centre of gravity of the hold of a ship, or of the space embraced by the outer surface of a ship's bottom. Let it it be proposed to find the distance of the centre of gravity of this space from the keel. We imagine it composed of several laminæ par-
Fig. 51, 52. allel to the section at the water's edge. The bulk of each lamina will be equal to half the sum of the two opposite surfaces of
Geom. 553. this lamina, multiplied by their perpendicular distance, and the centre of gravity will be at the same height in this lamina, as in the trapezoid $ABCD$, which is a section of this lamina, made by a vertical plane passing through the keel. We see, therefore, that the reasoning to be made use of here, in order to find the height GE, of the centre of gravity, is precisely the same as that in the last case, substituting only for *perpendicular* or *ordinate*, the word *section*; we have, therefore, this rule;

(1). *Take a sixth of the lowest section; a sixth of the highest, multiplied by triple the number of sections less 4; the second section from the lowest, double the third, triple the fourth, and so on; and call the result the first sum.*

(2). *Take half the sum of the lowest and highest sections, and all the sections between them, for the second sum.*

83. (3). *Divide the first sum by the second, and multiply the quotient by the common distance between two adjacent sections.*

We may make use of the same method in finding the distance
Fig. 51. of the centre of gravity from the vertical line XZ, drawn through a determinate point B of the stern-post, by imagining the bottom cut by planes parallel to the midship frame; but as it would be necessary to measure the surfaces of these sections, it is better to make use of those which have been already measured, in the last operation; accordingly we determine by the above method the centres of gravity G', G', of the several sections parallel to the keel. Their distances from the vertical XZ will be each the same as that of the centre of gravity G' of the corresponding lamina. We now multiply each section by the distance of its centre of gravity from the lines XZ, and regarding the several products as the ordinates of a curved line, like those in figure 50, we add the half sum of the two extreme products, to the sum of all the mean products, and divide the entire sum by the sum of all the

mean sections, plus half the sum of the two extreme sections, the common thickness of the laminæ being suppressed as a common 83.
factor to the dividend and divisor.

With respect to the centre of gravity of the vessel itself, whether laden or not, the investigation cannot be reduced to so simple a process. We must take into particular consideration the different parts which compose both the vessel and its lading. Having found the moments of these different parts with respect to a horizontal plane, supposed to pass through the keel; and the moments with respect to a vertical plane taken at pleasure perpendicular to the keel; we divide each of these two sums by the whole weight of the vessel, and we obtain the height of the centre of gravity, and its distance from the vertical plane with respect to which the moments were considered; and as it must also be in the vertical plane which passes through the keel, we shall have its position. But it may be remarked, that in the calculation of these moments, we must multiply, not the bulk of each part, but its weight, by the distance of the centre of gravity of this part; which centre is easily determined after all that has been said upon this subject.

XII

Properties of the Centre of Gravity.

115. It is evident from what we have said upon the subject of the centre of gravity, and upon the resultant of parallel forces, that if the parts of a body or system of bodies have each the same velocity, or tend to move with the same velocity, it is evident, I say, that the resultant of all these motions or tendencies would pass through the centre of gravity of the body or system, and that consequently the system would move, or tend to move, as if the several masses were all concentrated at the centre of gravity; and were together urged with a velocity equal to that which urges each of the parts.

116. We must infer reciprocally, that if any force be applied at the centre of gravity of a system of bodies; all the equal parts of this system will partake equally of this motion, and will all
proceed with an equal velocity, obtained by dividing the quantity 28.
of motion applied at this centre by the entire mass of the system,

and this velocity will have for its direction that of the force applied at the centre of gravity.

Indeed, whatever be the motions distributed among the parts of the system, we see clearly that they must have for a resultant the very force applied at the centre of gravity, since it is supposed that the system is free, and that there is consequently nothing to destroy any part of the force thus applied.

117. Also, since several forces applied at the same point, reduce themselves, by the preceding principles, to a single one, we infer generally, that *whatever be the number, direction, and magnitude of the forces which are applied at the centre of gravity of a system of bodies ;*

(1). *All parts of this system will have the same velocity ;*

(2). *This velocity will be in the direction of the resultant of all the applied forces ;*

(3). *It will be equal to the quantity of motion, which this resultant represents, divided by the entire mass of the system.*

118. Whence we conclude, that *while the forces which act upon a body, are capable of being reduced to a single one, the direction of which passes through the centre of gravity, this body will not turn about the centre of gravity.*

119. But if the forces which act upon a body cannot be reduced to a single one, or on the supposition that they admit of being so reduced, if the direction does not pass through the centre of gravity, all the parts of the system will not have a common motion. Nevertheless the centre of gravity will move in the same manner as if all the forces were applied directly at this point, as we now propose to show.

Fig. 53. 120. Let us in the first place suppose three bodies m, n, o, moving in parallel lines AA'', BB'', CC'', (situated in the same or in different planes,) and with velocities represented by the lines AA'', BB'', CC'', respectively, the motion of each being uniform. Let us suppose also, that G is the centre of gravity of these bodies, when they are in A, B, C; and that G'' is their centre of gravity, when they are in A'', B'', C'', where they will arrive in the same time, since their velocities are represented by AA'', BB'', CC'';

joining GG'', I say that this line will be parallel to AA'', BB'', &c., and that it will represent the course described by the centre of gravity during the supposed motion of the bodies m, n, o, and that it will be described uniformly.

(1). It is evident that the course described by the centre of gravity will be parallel to the lines AA'', BB'', &c.; for at whatever point we suppose it at any instant, if we imagine a plane passing through it, the sum of the moments with respect to this plane must be zero. Now if we conceive a plane parallel to the direc- 85.
tions of the bodies m, n, o, and passing through G, the moments with respect to this plane cannot but be zero during the whole motion, for the bodies in their motion are supposed not to alter their distances from this plane; their distances are therefore constantly the same, and consequently these moments are also constantly the same; but at the commencement of the motion, that is, when the centre of gravity is in G, the sum of the moments is zero; accordingly, it is still zero in whatever part of their directions the bodies are; the centre of gravity is consequently in a plane parallel to the directions of the bodies and passing through the first situation G of this centre. And, as in the reasoning here used, the position of this plane is not otherwise determinate than that it must be parallel to the directions of the bodies m, n, o, and pass through the point G; it may be shown, in like manner, that this centre is in any other plane parallel to the directions of the bodies and passing through the point G; it is consequently in the common intersection of these planes; therefore the centre of gravity moves according to GG'', parallel to the directions of the supposed bodies.

(2). The centre of gravity moves uniformly; that is, if when the bodies m, n, o, &c., have arrived at A', B', C', &c., we suppose that the centre of gravity is in G', we shall have

$$GG'' : GG' :: AA'' : AA' :: BB'' : BB' :: \text{\&c.};$$

in other words, the spaces described in the same time by the centre of gravity and the several given bodies will be as their velocities respectively.

Indeed, if we conceive a plane represented by ZX to which the directions of the several motions are perpendicular; we shall have, by the nature of the centre of gravity, 76.

$$m \times HA + n \times IB + o \times LC = (m + n + o) \times KG;$$

and for the same reason, when they are at A'', B'', C'',

$$m \times HA'' + n \times IB'' + o \times LC'' = (m + n + o) \times KG''.$$

If from the second of these equations, we subtract the first, bearing in mind that $HA'' - HA = AA''$, $IB'' - IB = BB''$, &c., we shall have

$$m \times AA'' + n \times BB'' - o \times CC'' = (m + n + o) \times GG'';$$

and for the same reason, when they are at A', B', C',

$$m \times AA' + n \times BB' - o \times CC' = (m + n + o) \times GG'.$$

Now since AA', BB', CC', are described uniformly in the
26. same time, these spaces must be as the velocities, AA'', BB'', CC''; consequently

$$AA'' : BB'' :: AA' : BB', \quad AA'' : CC'' :: AA' : CC',$$

which give $BB' = \dfrac{AA' \times BB''}{AA''}$, $CC' = \dfrac{AA' \times CC''}{AA''}$.

Substituting these values in the last of the above equations, we shall have

$$m \times AA' + n \times \frac{AA' \times BB''}{AA''} - o \times \frac{AA' \times CC''}{AA''} = (m + n + o) \times GG'$$

or, by making the denominator to disappear,

$$(m \times AA'' + n \times BB'' - o \times CC'') \times AA'$$
$$= (m + n + o) \times GG' \times AA''.$$

This equation divided by that in which GG'' enters, gives

$$AA' = \frac{GG' \times AA''}{GG''}, \text{ or } AA' \times GG'' = GG' \times AA'',$$

from which we have

$$GG'' : GG' :: AA'' : AA',$$

which was proposed to be demonstrated.

We remark that the equation in which GG'' enters, gives

$$GG'' = \frac{m \times AA'' + n \times BB'' - o \times CC''}{m + n + o}.$$

Now the lines AA'', BB'', CC'', GG'', are the velocities respectively of the bodies m, n, o, and of the centre of gravity G; consequently $m \times AA''$, $n \times BB''$, &c., are the quantities of motion respectively. Accordingly, since the reasoning we have pursued does not depend in any degree upon the number of bodies, we infer, as a general conclusion,

(1.) That, *if any number of bodies describe parallel lines, the centre of gravity describes a line parallel to them ;*

(2.) That *the velocity of the centre of gravity is equal to the sum of the quantities of motion of the bodies moving in one direction, minus the sum of the quantities of motion of those that move in the opposite direction, divided by the sum of the masses.*

121. If any one of the bodies be at rest, the velocity of this body will be zero, and the quantity of motion also will be zero. Thus it will disappear from the numerator of the fraction which expresses the velocity of the centre of gravity ; but the denominator, remaining unchanged, will in every case be the sum of all the masses.

122. If the sum of the quantities of motion of the bodies which move in one direction, be equal to the sum of the quantities of motion of those moving in the opposite direction, the numerator of the fraction which expresses the velocity of the centre of gravity will be zero. This centre of gravity, therefore, will be at rest. Accordingly, whatever be the parallel motions of several bodies, their common centre of gravity will remain at rest, when the sum of the quantities of motion of those that move in one direction is equal to the sum of the quantities of motion of those that move in the opposite direction.

123. Since the quantities of motion represent the forces ; and 28.
the resultant of any number of parallel forces is equal to the sum of those which act, or tend to act, in one direction, minus the sum of those which act, or tend to act, in the opposite direction ; we 50.
conclude, that, *if any number of parallel forces are applied to different parts of a system of bodies, the centre of gravity of this system will move as if the forces in question were all applied directly at this point.*

124. Let there be any number of bodies moving according to any given straight lines. If we imagine three rectangular coordinates, we may always decompose the velocity of each body into three other velocities, parallel respectively to these three lines. 73.
Now it follows from what we have just said, that the motion of the centre of gravity, in virtue of the motions parallel to one of

these lines, will be parallel to this same line; it will also be uniform, and equal to the sum of the quantities of motion (estimated parallel to this line), divided by the sum of the masses. If therefore we suppose that the motion of the centre of gravity, parallel to each of these lines, is thus determined, and that these three motions are afterwards reduced to one (which may be done, since they are all
72. applied at the same point), we shall have the course of the centre of gravity in a single line. Also, as the elements here employed, are simply the forces themselves which the bodies have parallel to the three co-ordinates, and as the single force of the centre of gravity is thus found to be composed of the resultant forces parallel respectively to these given lines, it cannot but be equal and parallel to the resultant of all the forces applied to the bodies in question; hence, *whatever be the directions and magnitudes of the forces applied to different parts of a system of bodies, the centre of gravity moves always, or tends to move, in the same manner, as if the forces in question were all applied directly at this point.*

125. In the foregoing article, we have said that we may always decompose the velocity of each body into three others, parallel respectively to three lines whose position is given. If the direction of one of the bodies, however, be parallel to the plane of two of the three assumed lines, or if it be parallel to one of these lines, it might seem that, in the first case, it would not admit of being decomposed, except into two forces, parallel to two of the three given lines; and that in the second case, no decomposition whatever could take place into forces parallel to the two other lines. Notwithstanding this apparent difficulty, the proposition is true universally. We
Fig. 54. see, for example, that so long as the line AB is not parallel to either of the lines XZ, XT, we can always decompose the force represented by AB into two others, AC, AD, parallel to these two lines respectively; but we perceive, at the same time, that the more AB approaches to a parallelism with XT, the more the force AD diminishes; so that it becomes zero, when AB is parallel to XT. There is not, therefore, in this case, the less propriety in supposing a decomposition into two forces, because one of them is zero. For a like reason, we may, in the same case, suppose a decomposition into three forces, parallel to three given lines XT, XZ, XY, two of which are equal respectively to zero.

126. From what we have now said, taken in connexion with that of article 122, we infer, that *the centre of gravity of a system of bodies will remain at rest, if, each of the forces applied to the several parts being decomposed into three other forces parallel respectively to three rectangular co-ordinates, the sum of the forces, or quantities of motion, parallel to each of these three lines be equal to zero*, the forces which act in opposite directions being taken with contrary signs.

127. When all the forces are in the same plane, it is evidently sufficient to decompose each force into two others parallel to two assumed lines, these lines being perpendicular to each other, and drawn in the same plane with the given forces; for the forces which are perpendicular to this plane being zero, the motion of the centre of gravity in virtue of these forces is also zero.

128. In all that we have said, we have supposed each of the bodies which compose the system, to obey fully and freely the force by which it is urged. But the same principles hold true no less when the bodies are constrained in their motions, provided the obstacles do not proceed from a force foreign to the system, that is, provided there are no impediments except those which arise from the difficulty of yielding to these motions by the manner in which they are disposed among themselves or connected with each other. This we propose to demonstrate after having first made known the general law of the equilibrium of bodies and the general law of their motion.

General Principle of the Equilibrium of Bodies.

129. *Whatever be the forces (acting or resisting), applied to a body, to a system of bodies, to a machine, &c., and whatever be the directions of these forces, if we conceive that each is decomposed into three others parallel respectively to three rectangular co-ordinates, it is necessary in order that all these forces should be in equilibrium, that the sum* of the forces which act parallel to each of these co-ordinates, should be equal to zero.*

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* By *sum of the forces* in what follows, is to be understood the sum of those which act in one direction, minus the sum of those which act in the opposite direction.

Indeed, whatever be the number and nature of the forces, we
have seen that they may always be reduced to three, the directions
73. of which are parallel to three rectangular co-ordinates. If there-
fore we suppose an equilibrium among all the forces of the system,
it is necessary that there should be an equilibrium among these
three resultants, or that each resultant should be equal to zero.
Now these resultants, being perpendicular to each other, can neither
35. increase nor diminish one another. But each is equal to the sum of
70. the partial forces parallel to it; therefore, there being an equilibrium,
the sums of the forces which by decomposition are found to act in
a direction parallel respectively to three rectangular co-ordinates
must each be equal to zero.

130. If all the forces are exerted in the same plane, the sum of each of the forces which by decomposition are found to be parallel respectively to two co-ordinates, drawn in this plane, will be zero. Moreover, if all the given forces should happen to be parallel to each other, the sum of their forces must be zero. These two cases are evidently comprehended in the general proposition.

131. It should be remarked, that this proposition holds true, whatever be the case in which the equilibrium occurs; but we should err by supposing that it is sufficient in order that an equilibrium may take place. The other conditions necessary for this effect vary according to the particular qualities or disposition of the parts of the system or machine in question.

132. The proposition, moreover, holds true, whether the forces which are applied to the different parts of the system are all active, or whether some are active, and others merely capable of resisting, as supports, fixed points, surfaces, &c., which oppose the action of forces; for impediments by destroying motion are equivalent in this respect to active forces.

D'Alembert's Principle, and concluding Deductions.

133. *Whatever be the manner in which several bodies come to change their existing state as to motion, if we conceive the motion which each body would have the following instant, on the supposition*

of its being free, as decomposed into two others, one of which is that which the body actually has after the change, the second must be such, that if each of the several bodies had had no other than this, they would have remained in equilibrium.

This proposition must be admitted, since, if the second motions be not such that an equilibrium would result from them in the system, the first component motions cannot be those that the bodies are considered as having after the change, for these would necessarily be altered by such a supposition.

134. Let us suppose now that several bodies, either free or connected together in any manner whatever, come to receive certain impulses which they cannot entirely obey on account of a reciprocal restraint, the centre of gravity will move as if all the bodies were free.

Indeed, whatever be the motion which each part of the system has, we may always conceive that which is impressed upon it as composed of two parts, namely, that which it actually takes, and a
second. But in virtue of these second motions, the system must be 40.
in equilibrium ; if we suppose, therefore, these second motions de- 133.
composed each into three others, parallel to three rectangular co-ordinates, the sum of the forces which would result from this decomposition, parallel to each of the three co-ordinates, must be
zero. Now the course which the centre of gravity tends to describe 129.
in virtue of each of these forces, is equal to the sum of the forces parallel respectively to each of the co-ordinates, divided by the
sum of the bodies. Consequently the course which it tends 120.
to describe in virtue of the changes arising in the system from the reciprocal action of the parts is zero ; accordingly the centre of gravity does not partake of these changes, that is, it moves as if each of the several parts of the system obeyed freely, and without loss, the force by which it is urged. Therefore, *the state of the centre of gravity of a body, or system of bodies, does not change by the reciprocal action of the parts of this body or system.*

135. Hence we infer; (1.) That, *if a body or system of bodies turn about its centre of gravity in any manner whatever, this centre*

of gravity will remain continually in the same state, as if the body did not turn.

Moreover, from this same principle and that of article 124 we conclude, that

136. (2.) If any body, whatever be its figure, or any assemblage of bodies, receive an impulse in any direction whatever as
Fig. 55. AB, which transmits itself entirely to the body; the centre of gravity G will move according to a line TS parallel to AB, in the same manner, as if this force were immediately applied at the centre of gravity in this direction. And if several forces act at the same time upon different points of this body, the centre of gravity will move as if all the forces in question were applied directly at this point.

137. If, therefore, at the instant the body receives an impulse in the direction AB, we apply at the centre of gravity in the opposite direction SG, a force equal to that which acts according to AB, the centre of gravity will remain at rest. Nevertheless it is evident that the other parts of the body would not remain at rest, since these two forces, although equal, are not directly opposite to each other. Now the only motion which this body can have, its centre of gravity remaining at rest, is evidently a motion of rotation about its centre of gravity.

Therefore, if a body receive one or several impulses, in directions which do not pass through the centre of gravity; (1.) This centre of gravity will move as if all the forces were applied directly at this point, each in a direction parallel to that which it actually has. (2.) The parts of this body will turn about their centre of gravity, as they would do by virtue of the forces which are actually applied to the body, if the centre of gravity were fixed.

138. We infer, moreover, that if the state of the centre of gravity of a body undergoes a change, this can proceed only from the action or resistance of new forces foreign to this body; and that consequently this change is always determined by seeking the resultant which all the forces would have, if they were applied to the centre of gravity, each in a direction parallel to that which it actually has.

XIII.

Application of the Principles of Equilibrium to the Machines usually denominated Mechanical Powers.

139. The general object of machines is to transmit the action of forces. The end to be attained is not always to augment the action of which the power employed is capable, when applied directly to the mass to be moved, or resistance to be overcome. Sometimes it is merely proposed to transmit this action in a determinate direction. At other times the purpose to be answered is to cause a body to describe spaces regulated upon certain conditions, relative either to time or other circumstances, conditions which do not always require that the force employed should augment as it is transmitted. We have examples of this kind of machinery in clocks, watches, orreries, &c.

The number and nature of the machines vary according to the object we have in view. But to be able to determine their effects, it is not necessary to consider them all separately. However compounded and varied they may be, they are merely combinations of a certain very limited number of simple machines. We come now to make known the properties of these simple machines. We shall afterward proceed to show how these properties are to be applied in estimating the effects of compound machines.

There are now usually reckoned seven simple machines, namely, *the rope machine*, *the lever*, *the pulley*, *the wheel and axle*, *the inclined plane*, *the screw*, *and the wedge*.

These machines, being considered simply with respect to a state of equilibrium, may be reduced to two, and indeed to one, namely, the lever. But in the case of motion, the nature of each leads to particular considerations, and requires a separate treatment.

Of the Rope Machine.

140. We proceed on the supposition that the ropes or cords employed are perfectly flexible. It will be shown, however, in the

sequel, what allowance is to be made for the want of this quality. Moreover, cords are first considered as destitute of weight, regard being had afterward to their gravity. The greater or less diameter of the cords also is not considered as affecting the communication of forces; since we may always substitute in imagination for these cords, considered as cylinders, a line or thread answering to their axes, the force employed being considered as acting by means of this thread only.

We employ cords to transmit the action of forces immediately, or in connexion with machines. But in order to judge of the effects of powers applied to machines by means of cords, it is necessary to ascertain the effects of which these powers are capable, when they act by means of cords alone.

Fig. 56. 141. Accordingly, let us consider three powers, p, q, r, as acting one against another, by means of three cords $A\,p$, $A\,q$, $A\,r$, united at A by a knot; and supposing the directions $A\,p$, $A\,q$, $A\,r$, to be known, we propose to determine the conditions necessary to an equilibrium among these forces, and the ratio of these forces.

(1.) It is evident, in the first place, that they must all three be in the same plane; for if one, the force r, for example, were not in the plane of the two others, we could always conceive it decomposed into two forces, one in this plane, and the other perpendicular to this plane, and consequently perpendicular to each of the two
40. forces p, q; this perpendicular force would not act, therefore, in any way against the forces p, q; and would accordingly have nothing to be opposed to it, and an equilibrium with respect to it could not take place.

(2.) These three forces being then in the same plane, it is necessary, in order that they may be in equilibrium, that some one of them, the force p for example, should produce two efforts, the one equal and opposite to the force q, and the other equal and opposite to the force r. Now if, after having produced rA, qA, we take any line AD to represent the force p, and upon AD as a diagonal we construct the parallelogram $ACDB$, the two sides AB, AC, will represent two forces, which acting conjointly according to these
40. directions, would produce the same effect as the force p. Accordingly, AB, AC, are the efforts that p actually opposes to the

two forces q and r; hence, in order that there may be an equilibrium, it is necessary that q should be represented by BA and r by CA, p by supposition being represented by AD. We have, therefore, the following proportions,

$$p : q :: AD : AB, \quad \text{and} \quad p : r :: AD : AC;$$

that is,

$$p : q : r :: AD : AB : AC.$$

Such is the ratio that must exist among the forces p, q, r, in order that an equilibrium may take place.

142. Since the two forces q, r, must be equal to the two forces AB, AC, which are the components of the force p, we infer that, when there is an equilibrium among three forces, any two of them must have the same ratio to the third, that two components have to their resultant.

143. Accordingly we have the proportion, 48.

$$p : q : r :: \sin BAC : \sin CAD : \sin DAB$$
$$:: \sin qAr : \sin rAS : \sin qAS,$$

pA being produced toward S; that is, *when three forces are in equilibrium, each is represented by the sine of the angle comprehended between the directions of the two others;* these directions being produced if necessary.

144. Since the three forces p, q, r, which are to be in equilibrium, are represented by AD, AB, AC, or, which amounts to the same thing, by the sides AD, AB, BD, of the triangle ABD, of which the angles ABD, BDA, DAB, are equal to the angles CAq, rAS, qAS, determined by the directions of the forces, it will be seen that all the questions which can occur with respect to the value and direction of the forces, requisite to an equilibrium, refer themselves to the subject of trigonometry. If, for example, the values of three forces p, q, r, were given, and it were proposed to find their direction, we should resolve the triangle DBA, the Trig. 38. three sides of which would be known, and the angles thus obtained would give the directions of the forces required. If we had given the two forces p, q, and the angle pAq, of their directions, or its supplement $qAS = DAB$; then we should have the two sides AB AD, and the contained angle DAB, from which we should readily

determine the side DB, or the force r, and the angle BDA, or its equal $r\,AS$, formed by the directions of r and p. If the angles
Trig. 35. formed by the directions of the three forces were given, we could not thence determine the absolute values of the three forces, but only their ratio to each other. In all other cases, the proposition
143. above established will be sufficient for a complete solution, when three things only are given.

145. If instead of having two forces, q and r, attached to two cords, these two cords were firmly fixed at q and r, or at any points respectively in their directions, AB, AC, would express the efforts supported by these points.

Fig. 56. Fig. 57. 146. We have supposed the three cords firmly attached by a knot A. But if the cord to which the power p is applied had a ring at its extremity A, through which the cord $q\,A\,r$ passed, we should not be able to assign the directions of the three cords. Indeed it is not sufficient, in this case, that the effort AB has the direction $q\,A$, and is equal to the force q, andt hat AC has the direction $r\,A$, and is equal to r; it is necessary, further, that the ring should not slip upon the cord $q\,A\,r$, which requires that the angle $q\,AS$ should be equal to $SA\,r$; that is, that the power p should be directed in such a manner, as to bisect the angle $q\,A\,r$. But we have always

$$p : q : r :: \sin q\,A\,r : \sin r\,AS : \sin q\,AS;$$

and as $r\,AS = q\,AS = \frac{1}{2}\,q\,A\,r$, this series of ratios becomes

$$p : q : r :: \sin q\,A\,r : \sin \tfrac{1}{2}\,q\,A\,r : \sin \tfrac{1}{2}\,q\,A\,r;$$

so that the two powers q and r are equal.

147. The same result would follow, if the cord $q\,A\,r$, drawn by the two powers r, q, passed over a fixed point A. The two powers r, q, must be equal, and the force exerted by them upon the fixed point will be directed in such a manner as to bisect the angle $q\,A\,r$, and its magnitude will be with respect to each of these two powers, as the sine of $q\,A\,r$ is to the sine of half $q\,A\,r$.

148. The foregoing articles being well understood, it will be easy to determine the conditions of equilibrium among as many powers as we choose to employ, applied to different cords, and united by the same or by different knots.

Let us suppose, in the first place, that each knot connects only three cords, and that they are all in the same plane, as represented in figure 58.

The power p is exerted against the two cords $A\,\varrho$, AB. Let the directions of these cords be produced; having taken AF to represent the power p, we form upon AF as a diagonal, and upon the prolongations AE, AD, as sides, the parallelogram $ADFE$. The force ϱ will be expressed by AE, and the tension of the cord BA by AD; so that, denoting by a this tension, we shall have

$$\begin{aligned} p : \varrho : a &:: AF : AE : AD \\ &:: \sin DAE : \sin FAD : \sin FAE \\ &:: \sin \varrho AD : \sin FAD : \sin FAE. \end{aligned}$$

Suppose the effort AD applied at B, according to BI, in the same straight line with AD, and equal to AD. The force BI is exerted against the power q, and against the cord BC. By producing, therefore, as above, the cords $q\,B$, CB, and forming the parallelogram $GBHI$, BH will represent the value of the force q, and BG the tension of the cord CB. We shall accordingly have, b denoting this tension,

$$a : q : b :: \sin GBH, \quad \sin IBG : \sin IBH.$$

Suppose the effort BG applied at C, according to CK, in a straight line with BG, and equal to BG. The force CK is exerted against ϖ and against r. If therefore we produce $r\,C$, $\varpi\,C$, and form as before the parallelogram $MCLK$, CM will express the value that must belong to the force r, and CL that which must be exerted by ϖ; whence,

$$b : r : \varpi :: \sin LCM : \sin KCL : \sin MCK.$$

If we would have immediately the ratio of the tension ϱ of any branch $\varrho\,A$ of the cord to the tension of any other branch, $C\,\varpi$, for example, it may be readily obtained in the following manner.

Of the series of ratios above found, if we take only those which relate to the tensions of the parts of the cord, $\varrho\,ABC\,\varpi$, we shall have

$$\begin{aligned} \varrho : a &:: \sin FAD : \sin FAE, \\ a : b &:: \sin GBH : \sin IBH, \\ b : \varpi &:: \sin LCM : \sin MCK; \end{aligned}$$

these being multiplied in order, we have

$\varrho : \varpi :: \sin FAD \sin GBH \sin LCM : \sin FAE \sin IBH \sin MCK.$

If we would have the ratio of the tension ϱ, to the tension b, we should multiply only the two first proportions. The other ratios may be found in a similar manner.

If it were proposed to determine the ratio of the powers among themselves, we have only to deduce from the above series of ratios the ratio of two consecutive powers to the tension of the same cord; thus,

$$\begin{aligned} p : a &:: \sin \varrho AD : \sin FAE \\ a : q &:: \sin GBH : \sin IBG \\ q : b &:: \sin IBG : \sin IBH \\ b : r &:: \sin LCM : \sin KCL. \end{aligned}$$

Taking the product of the corresponding terms, and reducing, we have

$p : r :: \sin \varrho AD \sin GBH \sin LCM : \sin FAE \sin IBH \sin KCL.$

To obtain the ratio of p to q, we should multiply only the terms of the two first proportions.

It will hence be seen how we ought to proceed when there is a greater number of powers, or when we would compare the tensions of the cords with the powers themselves.

149. If the powers p, q, r, bisect the angles ϱAB, ABC, $BC\varpi$, respectively, the angles DAF, FAE, would be equal; and the angles GBH, LCM, would have the same sines as the angles IBH, MCK, respectively; whence, by means of the above ratios, it will be seen that the different parts of the cord $\varrho ABC\varpi$ would be equally stretched.

150. If instead of the powers p, q, r, we substitute in A, B, C, fixed points or pivots, the pressure upon these points arising from the tension of the extreme parts of the cords, would be directed
147. in such a manner as to bisect each of the angles; and the tension of the several parts of the cords ϱA, AB, &c., would be equal. Accordingly, if two powers ϱ, ϖ, are exerted upon a cord passing
Fig. 59. round the periphery of a polygon or of any curve, the tension will communicate itself equally to every part, so that the two powers must be equal.

151. When the number of cords united by the same knot exceeds three, being in the same plane, or when, being in different planes, the number exceeds four, the directions being given, the ratios of the powers and of the tensions of the cords are not absolutely determinable; that is, if a certain number of powers (not less than those just stated) be in equilibrium according to known directions, we can substitute instead of them a like number of other powers directed in the same manner, but which, having very different ratios among themselves, are notwithstanding in equilibrium. If, for example, the four cords $A\ p$, $A\ q$, $A\ r$, $A\ \varpi$, are all in the same plane, and having taken AB to represent the force p, and having produced the cord $\varpi\ A$ to C, we suppose the effort AB composed of two others AC, AD, the first of which is equal and directly opposite to the power ϖ, nothing can be inferred from the direction AD of the action that is to oppose itself to the effort of the two powers q, r; nothing, I say, can be inferred from this direction, except that, produced, it must pass into the angle $q\,A\,r$; a condition which may evidently be satisfied in an infinite number of ways. Accordingly, if AD be drawn in any manner whatever, within the angle formed by $A\ q$ and $A\ r$ produced, and we construct upon AB as a diagonal, and upon the directions AC, AD, as sides, the parallelogram $ACBD$, and then upon AD as a diagonal, and upon $q\ A$, $r\ A$, produced, as sides, we construct also the parallelogram $AEDF$, AB being taken to represent the value of p, AC may be taken to represent that of ϖ, AF that of r, and AE that of q. This is evident, because the force AB is equivalent to the two forces AC, AD, the first of which, in order to be in equilibrium with ϖ, must be equal to ϖ, and the second AD is equivalent to the two forces AF, AE, which, to be in equilibrium with r and q, must be equal to r and q respectively. But it will be seen at the same time, that by giving to AD a different direction, AC, AF, AE, will have different values, but such notwithstanding that, being taken to represent the powers acting in these directions, an equilibrium would be produced; so that in this case, the directions remaining the same, there is an infinite variety of ways in which an equilibrium can be effected among the powers in question. Fig. 60.

152. The problem is of a similar character when the cords proceeding from the same knot, are in different planes, and amount

to more than four. But if the number does not exceed four, the directions being given, the ratios that must exist among the forces applied to these cords respectively, are determinate. For through
Fig. 61. any two of these cords, as $A\,p$, $A\,\varpi$, a plane may be supposed to pass, which produced would meet the plane $r\,A\,q$ of the two other cords in some line DAE, the position of which is determined by the directions of the four powers. Then, the direction $\varpi\,A$ being produced, and AB being taken to represent the power p, if upon AB as a diagonal, and upon the directions AD, AC, as sides, we construct the parallelogram $DACB$, AC will represent the value of the power ϖ, and AD the effort made by the power p against the two powers q and r acting conjointly. Accordingly, having produced $q\,A$ and $r\,A$ (which are in the same plane with AD) to F and G, if upon AD as a diagonal, and upon AF, AG, as sides, we construct the parallelogram $AFDG$, AF, AG, will represent the values belonging to the two powers q and r.

153. Finally, whatever the case may be, whether the cords are in the same plane or not, as a state of equilibrium requires that each knot should remain immovable, if the force or tension of each cord, applied to the same knot, be decomposed into three other forces parallel to three rectangular co-ordinates, it is necessary with respect to each knot that the sum of the forces parallel to each of these lines should be equal to zero; (it being well un-
129. derstood that by the word *sum*, as here used, is meant the sum of the forces that act in one direction, minus the sum of those which act in the opposite direction.) If the cords united by the same knot were in the same plane, it would be sufficient to decompose the tensions respectively into two forces parallel to two lines perpendicular to each other, and drawn in the same plane. This method would give in every case all the conditions of equilibrium, the cords being supposed to be firmly connected among themselves.

To give a simple example of this method, let it be proposed to find the ratio of three powers in equilibrium by means of three cords united by the same knot.

Fig. 62. Let us suppose for a moment that these three powers admit of being represented by the three lines AG, AB, AF, and in order to abridge the decomposition, let the two powers p, q, be

decomposed in the manner indicated by the figure, that is, each into two parts, one in the direction of p, and the other perpendicular to this direction. Then in the right-angled triangles BAC, FAI, we shall have, radius being unity,

$$\begin{aligned} BC = AD &= AB \sin q\,AC, \\ FI = AE &= AF \sin r\,AC, \\ AC &= AB \cos q\,AC, \\ AI &= AF \cos r\,AC. \end{aligned}$$ Trig. 30.

Therefore, according to the principle above referred to, we shall obtain, 129.

$$AB \sin q\,AC - AF \sin r\,AC = 0,$$

and $$AB \cos q\,AC + AF \cos r\,AC - AG = 0.$$

The first of these equations gives

$$AB \sin q\,AC = AF \sin r\,AC,$$

and consequently,

$$AB : AF :: \sin r\,AC : \sin q\,AC,$$

that is,

$$q : r :: \sin r\,AC : \sin q\,AC,$$

which agrees with what was before demonstrated. If the value 143. of AF be deduced from the first of the above equations, and substituted in the second, we shall have

$$AB \cos q\,AC + \frac{AB \sin q\,AC \cos r\,AC}{\sin r\,AC} - AG = 0,$$

or

$$AB \sin r\,AC \cos q\,AC + AB \sin q\,AC \cos r\,AC = AG \sin r\,AC.$$

But

$$\begin{aligned} \sin r\,AC \cos q\,AC + \sin q\,AC \cos r\,AC &= \sin (r\,AC + q\,AC), \\ &= \sin q\,A\,r; \end{aligned}$$ Trig. 11.

therefore,

$$AB \sin q\,A\,r = AG \sin r\,AC;$$

that is,

$$AB : AG :: \sin r\,AC : \sin q\,A\,r,$$

or,

$$q : p :: \sin r\,AC : \sin q\,A\,r,$$

143. which agrees also with the proposition above referred to.

154. We shall now inquire into the changes that take place in the communication of the action exerted by the powers in consequence of the gravity of the cords.

Let there be any number of powers applied to the same cord
Fig. 63. $\varrho\,ABC\,\varpi$ drawn at its two extremities by the two powers ϱ, ϖ, and retained at two fixed points ϱ and ϖ.

If we produce the two extreme cords $\varrho\,A$, $\varpi\,C$, until they meet in V, it is evident that the resultant of the particular tensions
43. of these two cords must pass through the point V. And since an equilibrium is supposed, the resultant of the three powers p, q, r, and of the tensions of the two intermediate cords AB, BC, must also pass through the point V; since, in order to an equilibrium, this resultant must be equal and directly opposite to the resultant of the tensions of the two cords $\varrho\,A$, $\varpi\,C$. But the resultant of the three powers, and of the tensions of the two intermediate cords, is nothing but the resultant of the three powers simply, because each of the two cords AB, BC, has by itself no action whatever, and consequently no effect upon any part of the system. Therefore the resultant of all the powers p, q, r, applied to the cord, passes through the point of meeting V of the two extreme cords.

42, &c. It has been shown how this resultant may be determined; but if the cords are parallel, as is the case when the powers p, q, r, are weights, since their resultant cannot but be parallel to them, its direction is found very simply by drawing through the point V a line parallel to one of the directions of these weights, that is by drawing a vertical or perpendicular line.

Accordingly, let there be any number of weights applied to the
Fig. 64. same cord $\varrho\,ABCD\,\varpi$ destitute of gravity. The two extreme cords being produced, and a vertical VX being drawn through their point of meeting, we can reduce the equilibrium of the whole system to the case in which the three powers, applied to three cords, are united by the knot V, and in which the power directed according to XVZ is the sum of the weights. We conclude, therefore,

that the tension ϱ is to the tension ϖ, as the sine $XV\varpi$ is to the sine of ϱVX. 143.

If a heavy cord now be considered as an infinite number of small weights uniformly distributed along the axis of this cord, it will be seen, that if ϖ represent the point where the power is applied Fig. 65. to the cord, and ϱ that in which this cord is attached to a machine, the action exerted by the power upon the point ϱ will be transmitted in the direction ϱV of a tangent to the curve representing the figure which the cord assumes by the action of gravity. This action is not equal to that of the power ϖ, except when the vertical, drawn through the point of meeting V of the two extreme tangents, bisects the angle $\varrho V \varpi$; and in general the action of the power ϖ, namely, that which it would transmit, if the cord were destitute of weight, is to that which it transmits in conjunction with the weight of the cord, as the sine of ϱVX is to the sine of ϖVX.

155. We remark, that strictly speaking, whatever force is employed to stretch a cord $\varrho\varpi$, this cord can never be made perfectly straight, except it be in a vertical position $\varrho\varpi'$. Let us suppose the cord $r A p$, destitute of gravity, to support the weight q, by means Fig. 66. of the two equal powers p, r, the directions of which are such as to form an angle approaching infinitely near to 180°, we shall have

$$q : p :: \sin CAD : \sin CAB;$$ 143.

or, DA being produced, Trig. 13.

$$q : p :: \sin CAS : \sin \tfrac{1}{2} CAD;$$

but the angle CAS is by supposition infinitely small, and $\frac{1}{2} CAD$ approaches infinitely near to a right angle; therefore q must be infinitely small with respect to p; and even where the weight q is infinitely small, the two parts of the cord still make an angle with each other, and are not, strictly speaking, in the same straight line.

It may hence be inferred, that a very small force q will cause a very great tension in the cords $A p$, $A r$, when the angle $r A p$ formed by them is very obtuse.

We are able, also, upon the same principle, to explain why, in blowing through a tube $A a$, into a flexible bag $a EBC a$, the ex- Fig. 67.

tremity B of which is attached to a weight p, we are able, I say, to explain, why a moderate impulse of the breath suffices to raise the weight p, although very considerable. Indeed, each half $a\,EB$, $a\,CB$, of the vertical section of this bag may be considered as a cord, pressed at each point by a perpendicular force equal to that exerted by the air. The resultant of all these pressures must be
154. directed according to FED, that is, it must pass through the point of meeting of the tangents belonging to the extremities of this cord, and must be to the effort made in the direction BD, as the sine of $a\,DB$ or of $a\,D\,u$, is to the sine of $FD\,a$. Now the angle $a\,D\,u$ is very small. Therefore a very small effort in the direction FD produces a very great effect in the direction BD; and accordingly the pressure exerted upon $a\,EB$ will cause a considerable effort in the direction BF, and the weight will be drawn by two forces of considerable magnitude in the direction BD, BF, which will have so much the greater effect, according as the angle FBD is smaller, since their resultant will approach so much the
50. nearer to the sum of the components.

Of the Lever.

156. By the *lever*, we understand an inflexible rod, of any fig-
Fig. 68, 69. ure whatever, so fixed at some point F, as to admit of no other motion, by the action of the forces that are applied to it, but a motion of *rotation*, that is, a motion, by which it turns about the fixed point F. This point is called the *fulcrum*.

We first consider the lever as an inflexible line without mass and without gravity. In the case of an equilibrium, we can easily make allowance for the gravity of the parts, by supposing it collected at the centre of gravity of this lever, and thus regarding it as a new force applied at this point according to a vertical direction. In case of motion, it is not at the centre of gravity that we are to suppose the mass collected, but at some other point to be determined hereafter.

We shall proceed on the supposition, that the forces applied to the lever, are all in the same plane with the fulcrum. We shall treat in another place of equilibrium and motion when the forces applied to the lever are in different planes.

157. Let there be two forces or powers p, q, applied at the two points B, D, of the lever, BFD, either immediately, or by means of two cords, or two rods without mass, acting upon this lever in the directions $B\,p$, $D\,q$, and being in equilibrium. It is proposed to determine the conditions of this equilibrium.

As one of the two powers, q for example, cannot be in equilibrium with the other except by means of the fulcrum F, it is evident that the power q must produce two efforts, one of which annihilates that of the power p, and the other is destroyed by the fulcrum F, and consequently passes through this point.

Let the lines $p\,B$, $q\,D$, representing the directions of the powers, be produced till they meet in some point A, and join AF. 43.
The power q may be supposed to be applied at A, according to $A\,q$; then if AG represent the value or magnitude of this power, and upon AG, as a diagonal, and in the directions AF, BAE, as contiguous sides, we construct the parallelogram $AHGE$; AE will represent the effort made by the power q, according to this line, and in a direction opposite to that of p; and AH will be that exerted against the fulcrum F. Indeed, although the point A is not connected with the two points B, F, the force q is distributed in the same manner as if A were thus connected. For it is evident, that if, without changing the forces or their directions, we connected the point A with the three points B, F, D, by means of three inflexible rods AB, AF, AD, without mass, this would not alter in any degree the supposed state of the system or the manner in which the force q is exerted. Now in this last case, the action of the force q would manifestly be communicated in the manner we have mentioned; therefore it would be communicated in the same manner, according to the first supposition. This being established, in order that there may be an equilibrium, it is necessary that the force AE should not only have a direction contrary to that of the force p, but that it should also be equal to p. As to the force AH, in order that it may be destroyed, it is sufficient that it be directed to the point F. Accordingly, if we designate the force exerted against the fulcrum by ϱ, we shall have

$$q : p : \varrho :: AG : AE : AH.$$

158. If from A towards B we take $AI = AE$, and join IH, $AIHG$ will be a parallelogram. But AI, AG, the sides of this parallelogram, represent the magnitudes and directions of the two forces p, q, consequently the diagonal AH represents their result-
38. ant; therefore, since AH thus represents the force exerted against the fulcrum, it may be inferred as a general conclusion that the force exerted against the fulcrum is precisely the resultant of the two forces applied to the lever; and that consequently these two forces act against the fulcrum as if they were immediately applied to it according to directions respectively parallel to those in which
144. they are actually exerted.

Indeed, this last truth may be rendered evident, by observing that, instead of the force q, may be substituted the two forces AE, AH, the first of which is destroyed by the force p, and the remaining force AH is the single effect to which the two forces p and q are reduced, and by consequence the resultant of these two forces.

159. By means of the ratios

$$q : p : \varrho :: AG : AE : AH,$$

15. above found, we are able to compare the forces q and p, as well with each other, as with the force exerted against the fulcrum. But as this ratio is not the most convenient, we proceed to find two others which may be employed for the same purpose.

48. (1.) According to a principle already established, we have

$$AG : AE : AH :: \sin HAE : \sin HAG : \sin GAE,$$

or,

$$:: \sin HAI : \sin HAG : \sin GAI,$$

since the angles HAE, GAE, have the same sines respectively as
Trig. 13. their supplements HAI, GAI; that is, the forces, q, p, ϱ, are each represented by the sine of the angle comprehended between the directions of the two others.

(2.) It has been shown that with respect to three forces of which one is the resultant of the two others, either two are always to each other reciprocally as the perpendiculars let fall upon their directions from any point taken in the direction of the third. Ac-
49. cordingly, if from any point in AF, as F, for example, we let fall

the perpendiculars FL, FM, upon the directions $p\,B$, $q\,D$, we shall have

$$q : p :: FL : FM.$$

In like manner, if from any point in the direction of the force q, as D for example, we let fall the perpendiculars DO, $D\,\varrho$, upon the directions of the force p, and ϱ, we shall have

$$p : \varrho :: D\,\varrho : DO.$$

The force q may also be compared in the same way with the force ϱ exerted against the fulcrum.

All these propositions hold true, whatever be the form of the lever, and whatever be the directions of the two powers employed.

160. When the directions of the two powers are parallel, in which case the resultant, or force exerted against the fulcrum, is parallel to them, the perpendiculars, let fall from the same point 50. in the direction of one of these forces, upon the directions of the other two respectively, are all in the same straight line LFM. Fig. 70. We may say, therefore, in this case, having drawn the line LFM perpendicular to the direction of the powers, that each force or power is represented by the part of this straight line comprehended between the directions of the two others.

161. If, moreover, the lever is straight, it will follow from the circumstance of the triangles FLB, FMD, being similar, that the parts FB, FD, BD, have the same ratio to each other as the parts FL, FM, LM; we may say, therefore, in this case that each force Geom. is represented by the part of the lever comprehended between the 202. directions of the two others.

Thus,

$$q : p :: FB : FD;$$

that is, the powers are to each other in the inverse ratio of the two arms of the lever FB, FD; so that the power q, in order to be in equilibrium, must be as much smaller than p, as the arm to which it is applied is longer than the arm to which p is applied. As to the force ϱ exerted against the fulcrum, it is equal to the sum of the two powers q, p; since these being represented by FD, FB, the former, or ϱ, is represented by BD.

Fig. 71, 72, 73. 162. If we make a distinction in the forces or powers, by regarding one, q for instance, as giving motion, p as receiving it, and F as a pivot or point of support, we may, in the manner of the ancients, make three sorts of levers, according to the three different situations in which the agent q can be placed with regard to p and F. Figure 71 represents what is called *a lever of the first kind*, in which the agent and the resistance are on opposite sides of the fulcrum, and the agent will have so much the more advantage according as its distance from the fulcrum is greater than that of the resistance. Figure 72 represents *a lever of the second kind*, in which the resistance is between the agent and the fulcrum, and which consequently is always favorable to the agent. Figure 73 represents a *lever of the third kind*, in which the agent is between the resistance and the fulcrum; in this case, therefore, the power of the agent is always employed to disadvantage; and such a lever is never to be used, where the object is to augment the effect of the agent; that is, where it is proposed to overcome a greater force. But as the purpose to be fulfilled is not always to increase the power of the agent, this circumstance does not prevent this third kind of lever being very usefully employed in machinery, where we would avail ourselves of every species of motion that we can dispose of. Thus in turning, in weaving, in spinning, and in various kinds of manufacture, where great velocity, and not great force is required, and where the hands of the laborer are occupied with the more important parts of the work, this species of lever is adopted with obvious advantage, the feet being employed to give motion to the machinery.

163. Before proceeding farther, we will observe, that setting aside friction, the fulcrum is not to be considered as simply a pivot,
Fig. 69. or support. Indeed, if the fulcrum F, instead of penetrating into the interior of the lever, as represented in the figure, only touched the surface, it is evident that, although the two powers q, p, were in the inverse ratio of the distances of the perpendiculars FM, FL, they would still not be in equilibrium, except in the single case, where the direction AF is perpendicular to BD (or to the tangent at F in figure 68) ; for, if AF were oblique, it would clearly tend to communicate motion to the lever in the direction BD. Thus, we should err in supposing, for example, that (friction and the gravity

of the lever being left out of the question) the two weights p and q, would remain in equilibrium in the inclined position represented, if, Fig. 74.
p being to q as $F q$ to $F p$, the surface of the lever merely rested upon the point F. The fulcrum, in order that there may be an equilibrium in all positions of the lever, must have the effect of a pin passing through it. In short, when we say, it is sufficient, that the resultant AF of the two powers should pass through the fulcrum F, it is taken for granted that the corresponding point F of the lever does not admit of any motion; for otherwise this condition is not sufficient. For example, if the lever BD were drawn by three forces p, q, r, applied to the three chords $B\,p$, $D\,q$, $F\,r$, there would Fig. 75.
not be an equilibrium, if AF were the direction of the resultant of p and q, although it should pass through the point F; it would be further necessary that the point of meeting A should be 64.
in $r\,F$.

164. Since the two forces p, q, in equilibrium by means of the lever BFD, must be in the inverse ratio of the perpendiculars FL, Fig. 68, 69.
FM; that is, since it is necessary that p should be to q, as FM to FL, it follows that $p \times FL = q \times FM$; in other words, the moments of the two forces, taken with respect to the fulcrum, or any other point in the direction AF, must be equal. 64.

165. As there cannot be a force without a tendency to motion, by the forces p, q, is to be understood the product of a certain mass by the velocities, that these forces would respectively communicate to this mass, if it were free. Thus let m be a certain mass, and u the velocity that the force p, acting freely, is capable of giving it; also, let n be another mass, and v the velocity that the force q is capable of giving it; in order that there may be an equilibrium, the following proportion is necessary; namely,

$$m \times u : n \times v :: FM : FL.$$

166. Let w be the velocity produced by the force of gravity in an instant; and let m, n, be two heavy bodies attached to two cords Fig. 76.
$BI\,m$, $DK\,n$, which passing over two round bodies I, K, transmit entirely to the lever BFD, according to any proposed directions BI, DK, the action of gravity of these bodies; we shall have $w\,m$, $w\,n$, as the measures of the forces with which these bodies act re-

27. spectively; we must have, therefore, in order that there may be an equilibrium,

$$wm : wn :: FE : FC,$$

that is,

$$m : n :: FE : FC;$$

therefore, in order that there may be an equilibrium between two bodies which are urged only with the force of gravity, or between any two bodies that tend to move with equal velocities, it is sufficient that the masses of these bodies be in the inverse ratio of the distances of their directions from the fulcrum.

167. But if the velocities with which the bodies tend to move, be unequal, it is not the masses, but the products of the masses into the velocities, which must be in the inverse ratio of the distances of their directions from the fulcrum.

168. If two finite and heavy masses m, n, are urged by finite and unequal velocities, according to the directions Im, Kn; as the velocity which gravity is capable of giving in an instant (or infinitely small portion of time) is infinitely small; in order that the two finite velocities may mutually destroy each other, it is sufficient that the quantities of motion which the two bodies would have in virtue of these velocities should be in the inverse ratio of FE, FC. But this equilibrium would not exist except for an instant; for when these velocities are mutually destroyed, the bodies m, n, subjected to the action of gravity, would receive quantities of motion, which would be in the simple ratio of the masses, and which consequently would no longer be in the inverse ratio of the distances FE, FC.

We hence see the difference between an equilibrium among bodies urged by gravity only, and an equilibrium among bodies urged by unequal finite velocities.

It may be remarked, moreover, that it is impossible to put in equilibrium a body urged by gravity only with a body urged by a finite velocity; and we may hence conclude, that if the
Fig. 71. weight p is in equilibrium with a force q exerted by an animal, this last does not tend to move the point D, except with a velocity infinitely small. If on the contrary the force q, applied at D,

acted by means of a blow or finite impression, it would raise the weight p, however great it might be, at least during a certain time, which, when p is very large, may be such that the eye cannot distinguish it; but the motion would not be the less real. This subject will be placed in a clearer light hereafter, when we come to treat of Collision.

169. By means of the ratio which we have established between the two powers p, q, and the force exerted against the fulcrum F, we shall be able to solve this general question. *Three of these six things, namely, the two powers, the force exerted against the fulcrum, and the three directions, being given, to find the three others.* When, however, the directions only are given, we can merely find the ratio of the forces p, q, ϱ. The solution in this case is evident from what has been said. It may be easily obtained also by geometrical construction upon which we will only observe, that when the directions are parallel, the question is solved by articles 53, 160; and that, in general, if it is proposed to determine the position of the fulcrum, when the powers p and q, and their position are known, the question reduces itself to finding, by article 38, the resultant of these two powers. 157. Fig. 68, &c. 144.

170. The problem is different when more than two powers are applied to the lever; in this case, as in that of the cords in article 151, we can vary without end the ratio or the directions of some of the powers, the others remaining the same, and yet not destroy the equilibrium. There is, however, this difference, between the lever and cords, that the condition of equilibrium in the former is single, whereas in the latter there are as many conditions of equilibrium as there are knots. It will suffice to point out the condition of equilibrium in the lever, when three powers are employed, to make it evident that the proposition will hold true, for any greater number of powers. 153.

171. Let the three powers p, q, r, directed according to $B\,p$, $E\,q$, $D\,r$, be in equilibrium by means of the lever BFD. The power q may be considered as exerted in part against each of the powers p and r,* and in part against the fulcrum F. Having Fig. 77.

* The power q cannot, strictly speaking, be considered as exerted against r, since they both tend to turn the lever in the same

produced the directions $B\,p$, $E\,q$, and taken from the point of meeting A, the line AH to represent the power q, we decompose this power into two others, one AG equal and directly opposite to the power p, and the other AC such as will admit of being in equilibrium with the power r, by means of the fulcrum F. Accordingly, if the direction $D\,r$ meet AC at the point I, we may suppose the force AC
33. applied at I, according to the direction $ACIL$; the force AC or IL must therefore be capable of being decomposed into two others, one IK equal and directly opposite to the power r, and the other IM directed against the fulcrum F. Thus the force q produces the three effects AG, IK, IM, of which the two first, being equal and directly opposite to the forces p and r, are destroyed, and the last, being directed against the fixed point F, cannot but be destroyed also. Now, since all the forces which act upon the lever, are p, q, r, or AG, IK, IM, and AG, IK are destroyed, we conclude that IM is the resultant of the three powers p, q, r, and that consequently the only condition necessary to an equilibrium is, that this resultant should pass through the fulcrum F. We see, therefore, that the powers p, q, r, act upon the fulcrum as if they were immediately applied to it according to directions parallel to those which they actually have; and this conclusion would hold true for any number whatever of powers, for we may always suppose one of the powers to be in equilibrium with all the others by means of the resistance of the fulcrum.

172. Since F must be in one of the points of the resultant, it must have the properties of which mention has already been
63. made; that is, *when several powers, exerted in the same plane, are in equilibrium by means of a lever of whatever figure, if from the fulcrum we let fall perpendiculars upon the directions of these forces, and multiply each force by the corresponding perpendicular, in other words, if we take the moments of the forces with respect to the fulcrum, the sum of the moments of the forces which tend to turn the lever in*

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direction about the fulcrum. The two powers p, q, being represented by AH, and GA or AG' respectively, and being exerted in these directions, are equivalent to AC or IL, the direction of which is opposed to r, since it tends to turn the lever in the contrary direction; and the resultant of IL and IK', that is, of p, q, r, is IM.

one direction, must be equal to the sum of the moments of those which tend to turn it in the opposite direction; which may be expressed generally, by taking with contrary signs the moments of the forces which tend to turn the lever in opposite directions, and saying, that *the sum of the moments must be zero.*

173. Accordingly, all that we have said with respect to the value and direction of the resultant, is applicable here to the deter- 53.
mination of the force exerted against the fulcrum, and the position of this point, whatever be the number of powers.

174. Knowing, for example, the two weights p and q, together Fig. 78.
with the length and weight BD of the lever, if we would determine the fulcrum F, upon which the whole would remain in equilibrium, we should consider the weight of the lever as a new force r, applied at the centre of gravity G of the lever, and it would be necessary that the moment of p with respect to the unknown point F should be equal to the sum of the moments of r and q, taken with respect to the same unknown point F.

Let the lever BD be straight, and of a uniform magnitude and specific gravity; and, bearing in mind, that on account of the directions of the forces being parallel, instead of the perpendiculars FL, FK, FM, we may employ the parts BF, FG, FD, which have the same ratio to each other, we shall have

$$p \times BF = r \times FG + q \times FD.$$

Let a be the length of the lever, x the distance BF; we shall have,

$$BG = \tfrac{1}{2}\, a, \qquad FG = \tfrac{1}{2}\, a - x, \qquad FD = a - x.$$

Let s be the specific gravity of the lever, or, in other words, the weight of each inch in length of this lever; a and x being also counted in inches; $\mathrm{s}\, a$ will be the whole weight r. We have accordingly,

$$\begin{aligned} p\, x &= \mathrm{s}\, a\, (\tfrac{1}{2}\, a - x) + q\, (a - x), \\ &= \tfrac{1}{2}\, \mathrm{s}\, a^2 - \mathrm{s}\, a\, x + q\, a - q\, x, \end{aligned}$$

from which we obtain

$$x = \frac{\tfrac{1}{2}\, \mathrm{s}\, a^2 + q\, a}{p + \mathrm{s}\, a + q}.$$

Let $a = 24$ inches, $p = 20$ pounds, $q = 4$ pounds, $s = \frac{1}{12}$ of a pound; we shall have,

$$x = \frac{\frac{1}{24}(24)^2 + 4.24}{20 + 2 + 4} = \frac{24 + 96}{26} = 4\tfrac{8}{13} \text{ inches};$$

that is, the fulcrum F, in order that there may be an equilibrium, must be placed $4\frac{8}{13}$ inches from the extremity B; whereas, by neglecting the weight of the lever, we should have

$$x = \frac{q\,a}{p + q} = \frac{96}{24} = 4 \text{ inches.}$$

If, on the other hand, the point B and the point F were given, and it were proposed to find the point D where the force q, supposed to be known as well as p, must be applied to produce an equilibrium; designating BF by b, and BD by y, the equation of the moments becomes

$$p\,b = s\,y\,(\tfrac{1}{2}\,y - b) + q\,(y - b)$$

$$= \tfrac{1}{2}\,s\,y^2 - s\,y\,b + q\,y - q\,b;$$

whence,

$$y^2 + \frac{(2\,q - 2\,s\,b)\,y}{s} = \frac{2\,p\,b + 2\,q\,b}{s}$$

and

Alg.109. $$y = \frac{s\,b - q}{s} \pm \sqrt{\frac{(q - s\,b)^2}{s^2} + \frac{2\,p\,b + 2\,q\,b}{s}},$$

$$= \frac{s\,b - q \pm \sqrt{(q - s\,b)^2 + (2\,p\,b + 2\,q\,b)\,s}}{s}.$$

The positive value of y in this result gives the distance BD in figure 78, and the negative value gives the distance BD in figure 79, the distance BF being supposed without gravity.

If we would have the distance or length y at which the weight of the part FD would of itself be sufficient to counterbalance the weight p, we should put $q = 0$, which reduces the above
Fig. 78. result to

$$y = \frac{s\,b + \sqrt{s^2\,b^2 + 2\,s\,p\,b}}{s}.$$

If, knowing p, q, BF, and the specific gravity of the lever DF, Fig. 80.
we would determine the distance FD at which the power q must be placed; designating FD by y, BF by b, we shall have $s\,y$ for the weight r; and, accordingly,

$$p\,b + \tfrac{1}{2}\,s\,y^2 = q\,y,$$

from which y is easily obtained.

In figure 78, it is evident that the longer the lever is, the more the power q is to be diminished, till it becomes zero, after which it must act in a contrary direction to produce an equilibrium.

In figure 80, as the lever is increased in length, the power q becomes at first less and less to a certain point beyond which it begins to augment. This may be easily shown in several ways, and among others by the equation

$$p\,b + \tfrac{1}{2}\,s\,y^2 = q\,y,$$

which gives

$$q = \frac{p\,b + \frac{1}{2}\,s\,y^2}{y},$$

by which it will be seen, that when $y = 0$, q must be infinite; and that when y is infinite, q must also be infinite. Accordingly, between these extremes the values of q, must be finite, and there must be some point where it will be the smallest possible. In order to determine this point, we have merely to put equal to Cal. 44.
zero the differential of the value of q, taken by regarding y only as variable; we have thus Cal. 11.

$$-\frac{(p\,b + \frac{1}{2}\,s\,y^2)\,d\,y}{y^2} + s\,d\,y = 0;$$

whence

$$s\,y^2 = p\,b + \tfrac{1}{2}\,s\,y^2,$$

and

$$y = \sqrt{\frac{2\,p\,b}{s}}.$$

Therefore the value of the smallest power q, which can be employed with a heavy lever, of the second kind, is

$$s\sqrt{\frac{2\,p\,b}{s}} \quad \text{or} \quad \sqrt{2\,p\,s\,b}$$

and the length of this lever is

$$\sqrt{\frac{2\,p\,b}{s}}.$$

It will hence be perceived, that when a weight is to be raised by a heavy lever, employed as in figure 81, a particular length is necessary in the lever, in order that the force may act to the greatest advantage, and that a given effect may be produced with the least possible force; and that a greater or less length would be attended with a loss of power. There is accordingly a difference in this respect between a heavy lever and a lever without weight.

Of the Pulley.

175. A *pulley* is a solid circle or wheel having a groove formed round its circumference, and an axis passing perpendicularly through its centre, and through a case or frame-work called the *block*. The several parts taken together, are sometimes called the block, and sometimes simply the pulley.

The different kinds of pulleys may be reduced to two, the *fixed*, and the *movable*.

Fig. 82, 83. The fixed pulley is that in which the power and the weight (or resistance to be overcome) are both applied according to directions that are tangents to the circumference of the pulley.

Fig. 84, 85, 86. In the movable pulley, the weight or resistance is applied at the centre, or in a direction passing through the centre or axis of the pulley.

This machine, considered in a general point of view, is susceptible of two sorts of motion; one by which the rope passing through the groove of the pulley, changes its place without altering the position of the body of the pulley, the other is such that the body of the pulley changes its situation at the same time. Thus a state of equilibrium requires two different conditions. The first is that the two parts of the rope which embraces the pulley should be equally

stretched, and thus mutually destroy each other. The second condition is derived from the first in the following manner;

176. From the tension of the two parts of the rope which passes over the pulley, there results an effort upon the body of the machine which may be determined by taking in the directions of the ropes, beginning at their point of meeting, IA, IB, equal to each other, and forming the parallelogram $IADB$, in which the diagonal ID represents the force exerted upon the body of the machine, IA being considered as representing the tension of the rope $O\,p$ or OG. Now since IT, IO, are tangents, and Fig. 82, 83, 84.

$$IB = IA,$$

it will be seen that ID produced would pass through the centre F of the pulley. Therefore, if the body of the pulley is not firmly fixed, ID cannot be destroyed, except the obstacle, whatever it be, which is to prevent the motion of the body of the pulley, is situated in some point of the line IF, extending from the centre F to the point of meeting of the two ropes. Thus, if the pulley is destined to turn in a block FG, fixed to some point G without, and admitting of a motion about G, an equilibrium will not take place except when the block has the direction FI. Fig. 83.

In like manner, if the body of the pulley, being embraced by a rope fixed to the point G, is movable, there will not be an equilibrium, except the effort applied at the centre F, or to the fixed block at this centre, is exerted in such a direction as to bisect the angle formed by the two parts of the rope OG, $T\,q$, and is at the same time to the tension of OG, $T\,q$, Fig. 84.

$$:: ID : IA : IB.$$

177. It is now easy to find the ratio of the tension of each part of the rope that passes round the pulley, to the force exerted upon the body of the pulley, and consequently to the force of which the movable pulley is capable. The tension of each part of the rope being represented by IA or its equal IB, the effort which is exerted upon the body of the pulley, will be expressed by ID. But in the triangle IAD. Fig 84.

$$IA : ID :: \sin IDA : \sin IAD,$$
$$\sin FI\,q : \sin OAD,$$
$$\sin FI\,q : \sin GI\,q.$$

We may say, therefore, universally, that *when there is an equilibrium by means of the simple pulley, fixed or movable; (1.) The tensions of the two parts of the rope which passes round the pulley, or the powers applied to them, are equal; (2.) That each of these powers is to the force exerted at the centre of the pulley, as the sine of half the angle formed by the two parts of the rope in question, is to the sine of the whole of this angle.*

Fig. 82, 83. Thus in the fixed pulley, there is no other advantage gained by the agent q, except that of being able to change at pleasure the direction in which the action shall be employed. But in the
Fig. 84, 85, 86. movable pulley, there is possessed by the agent q, the double advantage of a change of direction and an augmentation of the effect of the action. But it is to be remarked, that according as the direction is changed, the force exerted upon the centre varies, so that there is a direction in which the effect produced by a given power is the greatest possible; and this is when the two parts of the rope GO, $T\,q$, are parallel, as will be readily perceived.

Fig. 84. 178. If we draw the radii OF, FT, and the chord OT, the triangle OFT, having its sides perpendicular respectively to those
Geom. 209. of the triangle BID, will be similar to BID; whence

$$IB : ID :: FT : OT,$$

or

$$q : r :: FT : OT;$$

that is, *the tension of either part of the rope is to the force exerted against the centre F, as the radius of the pulley is to the chord of the arc embraced by the rope.*

Now it is evident that this last ratio is the greatest possible when the two parts of the rope are parallel; hence in the movable pulley the power is the least possible, or is exerted to the greatest advantage, when the two parts of the rope are parallel; and it is then half of the force exerted against the centre of the pulley. This second kind of pulley is made use of in tightening the sails

of a vessel, by attaching it to one of the corners as represented in figure 86.

179. If, therefore, the weight p is sustained by the power q, by means of several movable pulleys, embraced each by a rope, one extremity of which is attached to a fixed point, and the other to the block of a pulley, the ratio of the power to the weight will be that of the product of the radii of all the movable pulleys to the product of the chords of the arcs embraced by the ropes. Fig. 87.

Indeed if we call ϖ, ϱ, the forces exerted at the centres of the pulleys N, M, which are at the same time the tensions respectively of the two ropes attached to the centres of N and M; R, R', R'', being the radii, and c, c', c'', the chords of the arcs embraced by the ropes in the several pulleys N, M, L, we shall have 178.

$$\begin{aligned} q &: \varpi :: \mathrm{R} : c, \\ \varpi &: \varrho :: \mathrm{R}' : c', \\ \varrho &: p :: \mathrm{R}'' : c''; \end{aligned}$$

whence, by taking the product of the corresponding terms, Alg. 226.

$$q \varpi \varrho : \varpi \varrho p :: \mathrm{RR'R''} : c c' c'',$$

or

$$q \quad : \quad p \quad :: \mathrm{RR'R''} : c c' c'';$$

that is, when the cords are parallel, which gives

$$\begin{aligned} c = 2\,\mathrm{R}, \qquad c' &= 2\,\mathrm{R}', \qquad c'' = 2\,\mathrm{R}'', \\ q : p :: \mathrm{RR'R''} &: 2\,\mathrm{R} \times 2\,\mathrm{R}' \times 2\,\mathrm{R}'', \\ :: 1 \quad &: 2 \times 2 \times 2; \end{aligned}$$

in other words, the power is to the weight as unity to the number 2 raised to the power denoted by the number of movable pulleys. With three pulleys, for example, the power would sustain a force eight times as great.

180. But this arrangement of pulleys is not the most convenient. It is more common to employ one of the forms represented in figures 88, 89, 90, 91, 92, in which all the pulleys, both fixed and movable, are embraced by the same rope. Moreover all the fixed pulleys are attached to one block, and all the movable pulleys to another. Sometimes the centres are distributed upon different points of the same block as in figures 88, 89, 90, 91. Some-

times they are united upon the same axis as in figures 92, 93. The latter arrangement has the advantage of being more compact; but when a large number of pulleys are thus disposed in the same block, the power being applied on one side instead of being directed through the middle, the system is drawn awry, and part of the force employed is lost by the oblique manner in which it is exerted. This inconvenience does not belong to the pulleys represented in figures 88, 89, 90, 91; and in that represented by figure 94, the peculiar advantages of the two systems are united. Here two sets of pulleys having a common axis are attached to the movable block, and two to the fixed block, the inner set in each case being of a less diameter than the outer, so as to allow a free motion to the rope. Then the rope commencing at the middle of the upper block, after being made to pass over all the pulleys, will terminate also in the middle. This arrangement was invented by Smeaton.

181. But whatever difference there may be in this respect in the particular disposition of the pulleys, the ratio of the power to the weight may always be found by the following rule. *The power is to the weight as radius, or sine of* 90, *is to the sum of the sines of the angles made by the several ropes (meeting at the movable pulley) with the horizon.*

Fig. 88, 89. Indeed, if upon each of the ropes we take the equal parts IM, NP, &c., to represent the tension, and upon each of these lines, as a diagonal, we form a parallelogram, having one pair of its opposite sides vertical, and the other pair horizontal; instead of considering the weight p as sustained by the immediate tension of the ropes, we may regard it as supported by the horizontal forces IK, NO, &c., and
40. the vertical forces IL, NQ, &c. Now the first, being perpendicular to the action of the weight, contribute nothing to counterbalance this action; and in the case of an equilibrium these horizontal forces mutually destroy each other. The weight p, therefore, is wholly sustained by the resultant, that is, by the sum of the vertical forces IL, NQ, &c.; and, the ropes being all equally stretched, it is evident that q is to p as the tension of one of these ropes is to the entire sum of the vertical forces. But in the right-angled triangles IML, NQP, &c., we have

$$IM : IL :: 1 : \sin IML; \quad NP \text{ or } IM : NQ :: 1 : \sin NPQ;$$

the same may be said of the other ropes ; whence

$$IL = IM \sin IML; \quad NQ = IM \sin NPQ;$$

accordingly,

$$\begin{aligned} q : p :: IM &: IM \sin IML + IM \sin NPQ + \text{\&c.}, \\ :: 1 &: \quad \sin IML + \quad \sin NPQ + \text{\&c.} \end{aligned}$$

If the ropes are parallel and consequently vertical, the angles IML, NPQ, &c., will be right angles, and their sines will be each equal to radius, or 1. Therefore the power in this case will be to the weight as 1 is to the sum of so many units as there are ropes meeting at the movable pulley. Hence it will be seen, that *if one of the extremities of the rope is attached to the fixed pulley, the power will be to the weight as unity is to double the number of movable pulleys ; and if the extremity of the rope is attached to the movable pulley, the power will be to the weight as unity is to double the number of movable pulleys, plus* 1. Fig. 88, 90. Fig. 89, 91.

182. The general proposition above demonstrated holds true, whether the ropes are in the same plane or not ; and if the obstacle to be overcome be not a weight, that is, if the direction of the whole power of the pulley be not vertical, we have only to substitute for the angles which the ropes are supposed to make with the horizon, those which they would make with a plane perpendicular to the whole action of the pulley. In figure 93, for example, the power q is to the force exerted at G, as radius is to the sum of the sines of the angles made by the several ropes (meeting in CF) with a plane perpendicular to FG.

183. If several sets of pulleys are employed, it will be easy after what has been said to assign the ratio of the power to the weight. In figure 93, for example, the ropes being supposed parallel, the power q will be to the force exerted in the direction CB, 181.
as 1 is to 5. Now this last force performs the office of a power with respect to the system of pulleys BA, and accordingly is to the weight p, as 1 to 4. Therefore the power q is to the weight p, as 1×1 is to 4×5, that is, as 1 to 20.

184. In all that precedes, we have supposed the system of pulleys destitute of gravity and friction, and the ropes perfectly flexible. We shall see hereafter what allowance is to be made for friction

and the stiffness of the ropes. With respect to the gravity of the parts of the system which the power has to sustain, allowance is made for it, in the case of an equilibrium by adding it to the weight,
Fig. 90, 91. when its action coincides with that of the weight. But if, as in figure 93, the gravity of the system CF is not exerted in the same direction with the power q, BC, instead of being in the same direction with the power q, is in the direction of the resultant of the gravity of the system, and the force exerted independently of gravity.

Of the Wheel and Axle.

185. *The wheel and axle* consists in general of a grooved
Fig. 96. wheel, and a cylinder passing perpendicularly through the centre of the wheel, and resting at its extremities upon two fixed supports F, F. A power q, applied in the direction of a tangent to the circumference of the wheel, turns this wheel, together with the cylinder, which being firmly fixed to it, takes up successively the different parts of the cord $D\,p$ and with it the weight p, which it is proposed to elevate or draw toward the cylinder.

Fig. 96, 98, 99. Sometimes instead of a wheel, bars E, E, in the form of radii are the points at which the power is applied, and by which the same effect is produced. At other times the extremities of the cylinder
Fig. 97. are provided with winches q, q, at which the moving force is exerted.

When the axis of the cylinder is vertical the machine is called
Fig. 99, 100. a *capstan*. It is in this position that it is used on board of vessels, with this difference in the construction, however, that the figure of the axis is made conical instead of being cylindrical, that it may be worked more easily when the rope, having reached the lowest point, in turning to retrace its course would tend to check the motion.

186. But however the machine is placed, it will be seen that the action of the power and that of the weight which it is proposed to raise, are not exerted in the same plane, but in planes that are parallel or nearly so. The power produces two effects, one of which is exerted against the weight, and the other against the supports. In case of an equilibrium, these effects may be determined in the following manner.

The essential parts of the machine are represented in figure 101, where AMN is the plane of the wheel, FF the axis of the cylinder, and BDL a section of the cylinder, parallel to AMN, and passing through the cord $D\,p$.

Having drawn the radius EA to the point A, where the power q acts upon the wheel, suppose a plane FEA passing through FF, and EA, and meeting BDL in IB; IB will be parallel to EA. Join AB, and through this line and the direction $A\,q$ of the power, imagine a plane $q\,AG$ to pass meeting the axis FF in some point G. Lastly through B and G draw $B\,\varpi$ and $G\,r$ parallel each to $A\,q$.

This being supposed, the force q may be decomposed into two other forces ϖ, r, directed according to $B\,\varpi$, $G\,r$; and as this 55.
last passes through the axis of the cylinder, it can have no effect in turning the machine about this axis, and consequently can contribute nothing toward the support of the weight p. It will therefore be expended against the supports F, F. There will accordingly be only the force ϖ by which an equilibrium with the weight p is to be effected. Now (1.) This force is directed in the same plane BDL in which the action of the weight is exerted. (2.) The two lines $B\,\varpi$, BI, being parallel respectively to the two $A\,q$, AE, which are at right angles to each other, $B\,\varpi$ is perpendicular to BI, Geom.
and consequently a tangent to the circumference BDL. We may 70.
therefore consider BID as an angular lever, of which the fulcrum is at I; and since the distances of the directions of the two powers ϖ, p, from the fulcrum are equal, these two powers must be equal; we have accordingly $\varpi = p$. Let us now see what is the ratio of ϖ to q.

According to what has been laid down, we have

$$q : \varpi :: BG : AG;$$ 52.

but the similar triangles GBI, GAE, give

$$BG : AG :: BI : AE;$$

whence

$$q : \varpi \quad :: BI : AE;$$

or, since $\varpi = p$,

$$q : p \quad :: BI : AE;$$

that is, *in the wheel and axle the power is to the weight, as the radius of the cylinder to the radius of the wheel.*

Fig.102. 187. If the weight p be attached at some point B in the plane of the wheel, in such a manner, that the perpendicular IB upon its direction shall be equal to the radius of the cylinder, we may consider AIB as an angular lever the fulcrum of which is at the centre
159. I; and in order to an equilibrium, we must have

$$q : p :: BI : AI,$$

that is, the ratio between the power and the weight would be the same as the above. Therefore *the action of the power is transmitted to the weight by means of the wheel and axle, in the same manner as if the power and the weight were in the same plane.*

188. It is not the same, however, with respect to the force exerted against the supports. This varies according to the dis-
Fig. 101. tance of the plane BDL from the plane of the wheel. In order to determine what it is, we decompose the power q, considered as applied at E parallel to $A\,q$, into two forces parallel to $A\,q$,
55. and passing through F and F. We decompose likewise the power p, considered as applied at I, into two forces parallel to $p\,D$, and passing through F and F. By this means each support will be urged by two forces, the magnitude and directions of which will be known. It will be easy, therefore, to reduce these forces, in the case of each support, to a single one of a known magnitude and direction.

This method of finding the forces exerted against the two supports, is founded upon the fact, that the two forces ϖ and p reduce themselves to one which acts at I. If we conceive this decomposed into two forces parallel to the direction ϖ and p, and applied in I, they will have simply the values of ϖ and p. Accordingly, (1.) We may regard p as applied at I; (2.) The force ϖ, considered as applied at I, and the force r applied at G, cannot but have for a resultant the force q, by which they are produced, as we have
52. seen above; moreover this resultant passes through E, since

$$GI : GE :: GB : GA :: q : \varpi.$$

189. If the power, instead of being applied in the direction of
Fig. 99, 100. a tangent to the wheel, acted by means of the arms EE, and at

right angles to their length, the ratio of the power to the weight would always be found to be the same as above stated, by substituting for *radius of the wheel* the words *length of the arm;* this length being reckoned from the axis of the cylinder. But if the power acted in a direction not perpendicular to the arm IE, instead of the length of the arm we should take that of the perpendicular IR let fall upon the direction of the power; so that in this case the power will be to the weight as the radius of the cylinder to the perpendicular IR. Fig. 99.

190. Since $q : p :: BI : AE$, we have $q \times AE = p \times BI$; that is, the moment of the power is equal to the moment of the weight, these moments being taken with respect to the axis FF. If, therefore, several powers are employed at the same time, applied to different arms, the sum of the moments of these powers must be equal to the moment of the weight. Fig.101.

191. If the cord which supports the weight or which transmits the action of the power to the weight, were wound round a conical surface, or a surface of a variable diameter, instead of that of a cylinder, the ratio of the power to the weight, would also vary continually; and reciprocally, if the power, whose action is to be communicated through the medium of such a machine as that under consideration, varies continually, and is intended, notwithstanding, to produce the same effect, we arrive at the end proposed, by causing the action to be applied successively to radii that increase in length according as the power diminishes. We have an example of this adaptation of the machine to a varying power in watches and chronometers, in which the moving or *maintaining* power is a spring fixed at one of its extremities to the axis or *arbor* of a *barrel* Z, and which, after several revolutions or *coils*, is attached to the interior of this barrel. A *chain* with one of its extremities fixed to the convex surface of the barrel is wound round the conical axis or *fusee* Y, to which the other extremity of the chain is attached. As the spring uncoils, the barrel turns, and, drawing the chain, causes the fusee to turn; but since the force of the spring diminishes as it uncoils, a compensation is made for this reduction of the power, by giving a greater diameter to those parts of the fusee on which the last coils of the spring are exerted. By this contrivance, the machinery receives nearly equal impulses in equal times. Fig 103.

192. It would seem, therefore, by having regard only to an equilibrium, that we might diminish at pleasure the ratio of the power to the weight, and make a force, however small, counterbalance one however large, by means of the wheel and axle, and such machines as depend upon the same principle. But if we take into consideration their motion, and have respect also, as we must, to the nature of the agents to be employed, we cannot augment the effect at pleasure. The ratio of the radius of the cylinder to that of the wheel, is not arbitrary. It requires a particular adaptation to the purpose proposed, in order to produce the greatest possible effect.

Fig. 99. Suppose, for example, that the agent applied to the arm E, tends to move with a velocity u, and that the force of which it is capable, is $m\,u$, that is, equal to a known mass m urged with a velocity u. Let v be the velocity with which the point E would be moved in virtue of the resistance of p; then, if we call D the perpendicular distance of E from the axis, and δ that of p from the axis, we shall obtain the velocity that p would have, by the proportion,

$$D : \delta :: v : \frac{\delta\, v}{D},$$

since it is evident, that the point E and the point where the cord touches the cylinder, would have velocities proportional to their distances from the axis.

We must suppose, therefore, that at the instant when the power comes to exert itself, the velocity u is composed of the
133. velocity v, which actually takes place, and the velocity $u - v$, which is destroyed; and that at the same instant the weight p has the velocity $\frac{\delta\, v}{D}$, which actually takes place, and the velocity $\frac{\delta\, v}{D}$ in the contrary direction, which is destroyed; that is, the moving force $m\,(u - v)$ must be in equilibrium with the mass p, urged with the force $\frac{p\,\delta\, v}{D}$. Accordingly,

$$m\ (u - v) \times D = \frac{p\,\delta^2\, v}{D};$$

whence

$$v = \frac{m\, u\, D^2}{m\, D^2 + p\, \delta^2};$$

and the velocity of p, namely $\frac{\delta v}{D}$ will be

$$\frac{m u \delta D}{m D^2 + p \delta^2}.$$

Therefore, in order to know what ratio there must be between D and δ, in order that p may have the greatest velocity possible, it is necessary to put equal to zero the differential of this expression, taken by regarding δ only as variable; thus Cal. 48.

$$\frac{m u D d \delta (m D^2 + p \delta^2) - m u D \delta \times 2 p \delta d \delta = 0,}{(m D^2 + p \delta^2)^2}$$

or

$$m u D d \delta (m D^2 + p \delta^2) - m u D \delta \times 2 p \delta d \delta = 0,$$

whence

$$m D^2 + p \delta^2 - 2 p \delta^2 \quad \text{or} \quad m D^2 - p \delta^2 = 0,$$

which gives

$$\delta = \sqrt{\frac{m D^2}{p}} = D \sqrt{\frac{m}{p}}.$$

If, for example, the weight p be 100000lb., and the mass m or moving force be equivalent to a weight of 10lb., we shall have

$$\delta = D \sqrt{\frac{10}{100000}} = D \times \tfrac{1}{100};$$

that is, the radius of the cylinder must be a hundredth part of the arm IE, in order that the effect may be the greatest possible.

193. There are many machines which are referrible, either wholly or in part, to the wheel and axle, and consequently to the lever; such as *rack-work*, machinery in which wheels are connected by *bands*, *tooth and pinion* work, and instruments intended for drilling, boring, and screwing, although these last operations often depend in part upon another machine that remains to be described, namely, the inclined plane. In rack-work, the axis FE, having a *winch* $FR\,q$, carries a *pinion* the teeth or *leaves* of which act upon the toothed bar AB. The leaves of the pinion, in turning, raise the bar AB with a force which is to the force q applied to the winch, as the radius of the winch is to that of the pinion; and as the radius of the pinion is for the most part small compared with that of the winch, by the aid of such a Fig.105.

machine we are able to raise a very considerable weight with a moderate force.

194. Toothed wheels serve several purposes. Sometimes we employ them to augment a force, at others to increase a velocity, often to change the direction of a motion, and still more frequently to adapt motion to certain periods of time, or to render sensible certain motions or spaces, that the eye cannot distinguish.

Fig. 106. Several toothed wheels W, X, Y, Z, being connected together by the pinions w, x, y, z, it is proposed to find the ratio of the power q, applied to the first wheel, to the weight or effort p, sustained by the last pinion. Let D, D', D'', D''', be the greater distances or radii of the wheels, $\delta, \delta', \delta'', \delta'''$, the less distances or radii of the pinions. We shall consider the effort made by the leaf of any one of the pinions upon the tooth of the neighbouring wheel, as a power applied to this last; then E, E', E'', being
186. these efforts, we shall have

$$q : E :: \delta : D, \qquad E : E' :: \delta' : D',$$
$$E' : E'' : \delta'' : D'', \qquad E'' : p :: \delta''' : D''';$$

whence, by taking the products of the corresponding terms,

$$q : p :: \delta \cdot \delta' \cdot \delta'' \cdot \delta''' : D \cdot D' \cdot D'' \cdot D''';$$

that is, the power is to the weight as the product of the radii of all the pinions to the product of the radii of all the wheels.

If, for example, the radius of each pinion is one tenth of that of the corresponding wheel, we should have

$$q : p :: 1 : 10 \times 10 \times 10 \times 10,$$
$$1 : 10000;$$

that is, a power of one pound would counterbalance a weight of 10000 pounds.

What is gained, however, in point of force by the use of wheels and pinions, is lost in respect to velocity. Indeed while the wheel W turns once, the pinion w, turning in the same time,
Fig. 106. causes to pass only as many teeth of the wheel X, as it has leaves in its own circumference, so that if the wheel X has 48 teeth, and the pinion w six leaves, the wheel X would make only $\frac{6}{48}$ or $\frac{1}{8}$ part of a revolution, while W turns once round; it will hence be

seen that the wheel X goes just so much slower than W, Y so much slower than X, and so on.

195. From what is above said, it will be perceived how, by means of toothed wheels, the velocity may be augmented in any given ratio. Let there be, for example, the toothed wheel W, Fig. 107.
acting upon the pinion w; it is clear, that during one revolution of W, the pinion w will turn as many times as the number of leaves in the pinion is contained in the number of teeth of the wheel; that is, during one revolution of the wheel, the pinion will turn $\frac{N}{\nu}$ times, N denoting the number of teeth in the wheel, and ν the number of leaves in the pinion.

If therefore the axis of the pinion w carries a wheel, which acts also on a pinion x, we shall see that during one revolution of the wheel X, or of the pinion w, the pinion x will turn $\frac{N'}{\nu'}$ times, N' denoting the number of teeth in the wheel X, and ν' the number of leaves in the pinion x. Therefore while the wheel X makes a number of turns expressed by $\frac{N}{\nu}$, that is, during one revolution of the wheel W, the pinion x revolves a number of times expressed by $\frac{N'}{\nu'} \times \frac{N}{\nu}$ or $\frac{N'N}{\nu'\nu}$. And by reasoning in this manner for a greater number of wheels and pinions, it will be perceived that the number of times that the last pinion turns, during one revolution of the first wheel, is expressed by a fraction having for its numerator the product of the number of teeth in the several wheels, and for a denominator the product of the number of leaves in the several pinions.

When it is asked, therefore, what must be the number of teeth and leaves for a proposed number of wheels and pinions, in order that the velocity of the last piece shall be to that of the first in a given ratio, the question is indeterminate, that is, one which admits of several answers. Two examples will suffice to show how we ought to proceed in questions of this kind.

We will suppose that it is required to find how many teeth must be given to the two wheels W and X, and how many leaves to the pinions w and x, in order that the pinion x may make 50 revolutions while the wheel W makes one. We shall have

$$\frac{NN'}{\nu\nu'} = 50.$$

We know in this case only the quotient obtained by dividing NN' by $\nu\,\nu'$; we do not know either the dividend or the divisor. Let us take, therefore, arbitrarily for the divisor $\nu\,\nu'$ a number composed of two factors which shall be neither too small nor too great for the number of leaves to be allowed to the pinions. Suppose, for example, $\nu\,\nu' = 7 \times 8 = 56$, ν being 7, and ν' 8. We shall then have $\frac{NN'}{56} = 50$, or $NN' = 50 \times 56$. Now 50 and 56 not exceeding the number of teeth that can be given to the wheels W and X, I will suppose N to have 50; and consequently those of N' will be 56. If these two factors, or one of them, should happen to be too great, I should decompose them into their prime factors, and see if from the combination of these factors there would not result two smaller factors; or another number might be taken for $\nu\,\nu'$.

Suppose, for a second example, that it is proposed to find the number of teeth and leaves to be given to three wheels and their pinions, in order that while the last pinion turns once in twelve hours, the first wheel shall require a year to make one revolution.

The common year consisting of $365{,}25 \times 24 \times 60$ or 525949 minutes, and 12 hours being equal to 12×60 or 720 minutes, it is evident that during one revolution of the first wheel the last pinion will make a number of revolutions expressed by $\frac{525949}{720}$; we have, therefore, $\frac{NN'N''}{\nu\,\nu'\,\nu''} = \frac{525949}{720}$. Let us take arbitrarily $\nu = 7$, $\nu' = 8$; and we shall have $\frac{NN'N''}{7 \times 8\,\nu''} = \frac{525949}{720}$ or $NN'N'' = \frac{525949}{720} \times 7 \times 8\,\nu'' = \frac{3681643\,\nu''}{90}$.

Of the Inclined Plane.

Fig. 108. 196. If a body p of any figure whatever, touching a plane XZ in any point C, is urged by a single force, it can remain at rest on this plane only when the direction of this force is perpendicular to the plane, and is such at the same time as to pass through

the point C. The necessity of the first condition is evident. As to the second, it will be seen, with a moment's attention, that this is 38. not the less necessary; since, if the direction AD of the body p', for example, although perpendicular to the plane, does not pass through the point of contact C', the resistance of the plane, which cannot be exerted except according to the perpendicular at C, would not be directly opposed to the force AD, and consequently would not destroy it, even when it is supposed equal to this 137. force.

197. If the body, instead of touching the plane only in one point, touches it in several points, it is not indispensable that the Fig.109, 110. single force AD, which acts upon it, should pass through any one of these points; but it is necessary that it should be perpendicular to the plane, and that it should be capable of being decomposed into as many forces perpendicular to the plane, as there are points which rest upon it, and that they should be such as to pass through these points. Thus if the body p, for example, were in contact with the plane at the points C, C', and the force AD Fig.109. were not in the plane which passes through the two perpendiculars raised at the points C, C', an equilibrium would not take place, because the force AD could not be decomposed into forces passing through C and C', without a third arising which would not be counterbalanced.

198. Hence, if a body which touches a plane in one or in several points, be urged by several forces directed at pleasure, it is necessary, (1.) That these forces should admit of being reduced to a single one perpendicular to the plane; (2.) That this, in the case where it does not pass through one of the points of contact, should be capable of being decomposed into as many forces parallel to it, as there are points of contact, and that these should pass each through one of the points of contact.

199. If the single force which urges a body be gravity, it is necessary that the plane should be horizontal; and if the vertical plane, drawn through the centre of gravity of the body, do not pass through one of the points of contact, it is necessary, at least, that it should not leave all the touching points on the same side.

200. If therefore, the body be urged only by two forces, it is necessary; (1.) That the two forces should be in the same plane; (2.) That this plane should be perpendicular to that on which the body rests; (3.) That the resultant (which must be always perpendicular to this last plane) should not leave all the points of contact on the same side; and if one of these forces be gravity, it is necessary, moreover, that this plane should be vertical and pass through the centre of gravity of the body.

201. Let us now see what ratio must exist between two forces which hold a body in equilibrium upon a plane. Let $F\,q$, $F\,p$,
Fig. 111. be the directions of these two forces, and AB the intersection of the plane of these forces with that upon which the body rests; having drawn the perpendicular FH upon AB, let us suppose that on this line, as a diagonal, and upon $F\,q$, $F\,p$, as sides, the parallelogram $FEDC$ is constructed. In order that the resultant of the two forces q and p may be directed according to FD or FH, it is necessary that the two forces q and p should be to each other as FC to FE; and then the two forces p and q, and the pressure which they exert upon the plane, and which I shall represent by ϱ, will be such as to give the proportion

38. $$q : p : \varrho :: FC : FE : FD.$$

202. According to article 48, we have likewise

$$q : p : \varrho :: \sin EFD : \sin CFD : \sin EFC.$$

203. From the two points A, B, taken arbitrarily in AB, we let fall upon the directions of the two forces q, p, the perpendiculars AG, BG. The triangle ABG having its sides perpendicular respectively to those of the triangle FDE, the two triangles will be similar; hence

$$AG : BG : AB :: DE \quad \text{or} \quad FC : FE : FD :: q : p : \varrho;$$

accordingly

$$AG : BG : AB :: q : p : \varrho.$$

But

Trig. 32. $$AG : BG : AB :: \sin ABG : \sin BAG : \sin AGB,$$

therefore

$$q : p : \varrho :: \sin ABG : \sin BAG : \sin AGB;$$

that is, when two forces only act upon a body to retain it in equilibrium upon a plane; if we imagine two other planes to which the forces are perpendicular, these two forces and the pressure upon the given plane, are represented each by the sine of the angle comprehended between the planes to which the two other forces are perpendicular.

204. Since the ratios which we have established, take place whatever be the nature of the two forces p and q, they will hold true when one of the forces, p for example, is gravity; in this case the plane BG is horizontal, and the intersection BG is called the *base*, and AL, perpendicular to BG, the *height* of the plane. Fig.112.

205. Since by article 202,

$$q : p : \varrho :: \sin EFD : \sin CFD : \sin EFC,$$

we have

$$q : p :: \sin EFD : \sin CFD,$$
$$:: \sin HF\,p : \sin HF\,q;$$

if, therefore, knowing the weight p, the power q, and the angle $HF\,p$, which the direction of the weight p makes with the perpendicular to the plane, we would determine the angle which the direction of the power q must make with the same perpendicular, we shall obtain it by the above proportion, which gives

$$\sin HF\,q = \frac{p \times \sin HF\,p}{q}.$$

But when an angle is determined by its sine, there is no reason for taking as the value of this angle, the angle itself found in the tables, rather than its supplement. Accordingly, the same weight Trig. 13.
may be supported upon the same plane, by the same power, directed in two different ways. These two directions must therefore be such that the two angles $HF\,q$, $HF\,q$, which they form with the perpendicular FH, may be supplements to each other. Now if we produce the perpendicular HF, toward I, the greater of these two angles $HF\,q$ is the supplement of $q\,FI$; therefore, since it must also be the supplement of the smaller angle $HF\,q$, it follows that $q\,FI$ is equal to the smaller angle $HF\,q$. Hence

the two directions according to which the same power will sustain a given weight upon the same plane, are equally inclined with respect to a perpendicular to this plane, and consequently with respect to this plane itself; and they both fall on the side of a perpendicular to this plane, opposite to that in which the gravity of the body is directed.

206. In the same proportion,

$$q : p :: \sin HF\,p : \sin HF\,q,$$

if, instead of the angle $HF\,p$, we put the inclination ABG of the Geom. 209. plane, which is equal to this angle, and instead of $\sin HF\,q$, its equal $\cos A'F\,q$, FA' being drawn parallel to BA, we shall have

$$q : p :: \sin ABG : \cos A'F\,q,$$

and hence

$$q = \frac{p \times \sin ABG}{\cos A'F\,q}.$$

Therefore, the inclination of the plane and the weight remaining the same, the power q must be so much the smaller, as the cosine of its inclination to the plane is greater; accordingly, as the Fig. 113. greatest of all the cosines is that of $0°$, we say that *the direction in which a power acts to the greatest advantage, in sustaining a weight upon an inclined plane, is that which is parallel to this plane.*

207. In this case the proportion

$$q : p :: \sin ABG : \cos A'F\,q$$

becomes

$$q : p :: \sin ABG : 1 \text{ or radius.}$$

Fig. 113. Now if, from the point A, we let fall the perpendicular AL upon the horizontal line BG, we shall have in the right-angled triangle ALB,

Trig. 30.

$$\sin ABG : 1 :: AL : AB;$$

therefore

$$q : p :: AL : AB;$$

that is, *when the power acts in a direction parallel to the plane, it is to the weight as the height of the plane is to its length.*

208. If the direction of the power be horizontal, the angle $A'F\,q$, being the complement of BAL, the proportion becomes Fig.114.

$$q : p :: \sin ABG : \cos A'F\,q,$$
$$q : p :: \sin ABG : \sin BAL,$$
$$:: AL : BL;$$ Trig. 32.

that is, *when the direction of the power is parallel to the base of the inclined plane, the power is to the weight as the height of the plane to its base.*

From the proportion

$$q : p :: \sin ABG : \cos A'F\,q,$$

we infer, as a general conclusion, that so much less power is required according as the inclination of the plane is less, and according also as the inclination of the power to the plane is less.

We have said nothing of the point where the direction of the power is to be applied to the body. This point is determined only by the condition, that the direction of the power meet the vertical drawn through the centre of gravity of the body in a point from which a perpendicular let fall upon the plane has the conditions mentioned in article 196, &c.

We hence see that a homogeneous sphere cannot be sustained upon an inclined plane, except when the direction of the sustaining force passes through the centre of the figure, which is at the same time the centre of gravity.

209. If several powers, instead of one, are opposed to the action of the weight, what we have said respecting the power q, is to be understood of the resultant of these several powers. If the body p, for example, is supported upon an inclined plane by the Fig.115. combined action of a power q, and of the resistance of a fixed point B, to which is attached the cord $HD\,q$, passing round the body; through the point of meeting S, of the two cords BH, $q\,D$, suppose a line SF drawn so as to bisect the angle formed by the cords. If this line cut a vertical line passing through the centre of gravity in a point F, from which a perpendicular can be let fall upon the plane that shall pass through the point of contact

H, the equilibrium will be possible; and the ratio of the weight p to the effort in the direction SF will be determined by the foregoing rules. The ratio of the effort, in the direction SF, to the power q, will be the same as in the movable pulley. Thus if
207. the power q is exerted in a direction parallel to the plane, the weight p will be to the power q, as the length of the plane to half its height; that is, the power will be only one half of what would be necessary without the aid of the pulley, or fixed point B.

210. With respect to the whole pressure exerted upon the plane, it will be easily determined by the ratios above established. As to the particular pressure, however, that takes place upon each of the points where the body rests upon the plane, it is absolutely indeterminate, except in the case where the body touches only in two points; and in this case the whole pressure
161. is divided between these two points in the inverse ratio of the distances of its direction from these points. In every other case there are no other conditions for determining the several pressures except (1.) That the sum of them must be equal to the whole pressure. (2.) That the sum of their moments, taken with respect to an axis perpendicular to the direction of the whole pressure, is zero; the same will be true of the sum of the moments with respect to another axis perpendicular to the first. These two axes, moreover, pass through a point in the direction of the whole pressure. Thus, when a body rests upon a plane by means of a plane surface, there is no reason for supposing that all the points upon which it rests should experience equal pressures, except when it has the figure of a right prism or a right cylinder.

211. With respect to bodies which rest upon several planes at once, either in virtue of a single force, or of several forces, in which we comprehend their gravity, the general law of equilibrium is, (1.) That the resultant of all these forces must admit of being decomposed into as many forces as there are points on which the body rests; (2.) That these must be perpendicular to the plane touching the body at this point.

Let a heavy body KGI be placed in equilibrium upon two
Fig. 116. inclined planes; this state can continue only while the weight of the body is destroyed by the resistance of the planes; if there-

fore the body is in contact with each of the planes only in a single point, and perpendiculars IO, KO, be drawn through these points, they must meet in some common point O, of the vertical passing through the centre of gravity G, in order that the weight of the body may admit of being decomposed into two other forces having directions perpendicular to these planes. The components IO, KO, will represent the pressures exerted upon the planes. It hence results, that *the plane which passes through the points of support and the centre of gravity must be vertical*, or *perpendicular to the inclined planes*, or *to their common intersection, which will consequently be horizontal.*

What is here said is not peculiar to the case of a body urged by gravity simply. Whatever be the forces acting at I, K, their resultant must conform to what we have said of the vertical passing through the centre of gravity.

Let XZ be a horizontal plane passing through the intersection B of the inclined planes; and through the point K, draw KH also horizontal; and let the weight of the body KIG be represented by ϱ, and the pressures exerted upon the two planes AB, BC, by p, q, respectively. In order to obtain these pressures, we must suppose the weight ϱ of the body to be a vertical force applied at O; thus regarded, it may be decomposed into two others, directed according to OI, OK; we have accordingly the following proportions,

$$\varrho : p : q :: \sin IOK : \sin GOK : \sin IOG,$$ 48.

or, since the angle $CBZ = GOK$, and $ABX = IOG$, and the angles IBK, IOK, are supplements of each other, Geom. 80.

$$\varrho : p : q :: \sin ABC : \sin CBZ : \sin ABX,$$ Trig. 13.

$$:: \sin HBK : \sin BKH : \sin KHB,$$

$$HK : HB : BK.$$ Trig. 32.

212. These principles are sufficient for determining, under all circumstances, the conditions of equilibrium, where planes are concerned. By means of them we are enabled to explain the strength of arches, and in general why hollow bodies, whose exterior surface is convex, are better fitted, on this account, to resist a compressing force. If, for example, a body is composed

Fig. 116. of four parts $ABCD$, $CDFE$, $FEGH$, $ABGH$, the exterior and interior surfaces of which are circular and concentric, and the same force be applied to the centre of gravity of each part, and be directed toward the common centre of the whole, no separation can take place among the parts, however great the force employed, provided the material itself be sufficiently hard. For it will be seen, that the force belonging to each part may be considered as decomposed into two others perpendicular respectively to the two plane faces of this part, and that consequently between each pair of contiguous planes there will be two equal and directly opposite forces; so that the several forces will mutually destroy each other, and a general equilibrium will be the result. The parts $ABCD$, &c., are called *voussoirs*. In a regular arch, the upper voussoir is distinguished by the name of *key-stone*. The surfaces which separate the voussoirs are technically termed *joints*. The interior curve of the arch is called the *intrados*, and the exterior, or that which limits all the voussoirs, when they are in equilibrium, is called the *extrados;* the masses of masonry at each end, that support the arch, are the *abutments*. The beginning of the arch is called the *spring*, the middle the *crown*, and the parts between the spring and the crown, the *haunches* of the arch. The part of the abutment from which the arch springs, is termed the *impost;* and the distance between the imposts the *span* of the arch.

Of the Screw.

Fig. 117, 118. 213. The *screw* AB, is a solid cylinder having a protuberance or *thread* raised upon its convex surface, and carried round obliquely, and continually with the same inclination to the axis.

The *nut* is a hollow cylinder with a spiral groove cut upon the concave surface, and fitted to receive the thread of the screw. The former is sometimes called the *external*, and the latter the *internal* screw.

Sometimes the nut is fixed, and the screw in turning has all its threads carried successively through it; sometimes the screw is fixed, and the nut in turning passes the whole length of the screw. In each case, while the power is applied at the same distance from the axis of the screw, there is always the same ratio between this

power and the force which it is capable of exerting in the direction of the axis.

214. We shall have a pretty just idea of the screw, by representing the thread as formed by wrapping round the cylinder the hypothenuses CK of as many right-angled triangles CIK, as Fig. 119. there are revolutions of the thread, each triangle having for its height, the distance CI between two adjacent threads, and for its base IK, the circumference of the cylinder corresponding to the point I; so that, according as the thread becomes thicker, IK is increased in length, the height CI remaining the same.

In figure 118, where the threads are edge-shaped, according as the protuberant part becomes thicker, or departs farther from the axis, we must suppose that the base IK increases, and that the Fig. 119. height CI diminishes.

215. The screw AB being fixed, and having a vertical position, no allowance being made for friction, or for the nut having its natural gravity, it is evident that the nut in turning would Fig. 117, 118. pass over the several threads of the screw by sliding upon each as upon an inclined surface. It is also evident that this tendency may be overcome by applying to the nut XZ, a certain power, which admits of being directed in several different ways. But as the nut has manifestly no motion, if it be prevented from turning, we shall confine ourselves to inquiring what must be the ratio between the weight of the nut, or in general between the force which urges it in a direction parallel to the axis of the screw, and the force capable of preventing its turning. Of the several points of the nut, we shall first consider only that which rests upon one point of the thread of the screw.

The force which acts immediately on this point to prevent the turning, and that which tends to make it descend parallel to the axis, must be regarded as being in equilibrium upon an inclined plane whose height is the perpendicular distance between two adjacent threads, and whose base is the circumference of the circle which would be described by the point in question. This follows from what we have said of the nature of the screw. Now of these two forces, the first is parallel to the base of the inclined plane, and the second is perpendicular to it; hence, by article 208, it will

be seen that the part of the force parallel to the axis of the screw which is exerted upon any point of the thread, is to the force which it is necessary to apply at this point to prevent the turning, as the base of the inclined plane is to its height, that is, as the circumference of the circle, which would be described by the point of application, is to the perpendicular distance between two adjacent threads. Therefore, if we call p the first force, and q' the second, and δ the distance of the point of application from the axis, h the height of the supposed inclined plane or perpendicular distance between two adjacent threads, and 2π the ratio of the radius of a circle to its circumference, we shall have

$$p : q' :: 2\delta\pi : h.$$

But each point of the nut is not supported directly; the whole is subjected to a certain power q, applied at some point of the nut whose distance from the axis may be represented by D. It is hence evident, that D being greater than δ, there will be necessary for each point, a force so much the less, according as the distance D is greater; so that if we call q the part of this force which at the distance D is capable of the same effort as q' is at the distance δ, we shall have

$$q' : q :: D : \delta.$$

Multiplying this proportion by the former, we shall have

$$p : q :: 2\pi D\delta : h\delta :: 2\pi D : h;$$

that is, for each point of the nut that rests upon the thread of the screw, there is the same ratio between the force exerted parallel to the axis, and that which at a given distance D prevents the turning; and this ratio is that of $2\pi D$ to h. Now $2\pi D$ is the circumference of the circle which would be described by the power q in turning; we conclude, therefore, that the sum of all the forces p which urge the nut parallel to the axis, is to the sum of all the powers q necessary to prevent the turning, as the circumference of the circle which would be described by the power q, is to the distance between two adjacent threads of the screw.

216. Hence the force which it is necessary to employ parallel to the axis of the screw, to prevent the power q from turning the nut, must be to this power q, as the circumference which this

power tends to describe, is to the distance between two adjacent threads.

217. Therefore, upon the same screw, the effect of the power q will be so much the more considerable according as it is applied at a greater distance from the axis; and upon different screws, the power being applied at the same distance from the axis, the effect will be so much the more considerable according as the distance between the threads is less.

218. The screw, therefore, is a compound machine partaking of the inclined plane and the lever; it is advantageously employed in compressing bodies and for several other purposes. Friction, however, greatly modifies the effect which this machine ought to produce according to the ratio above established.

219. In order that the nut may pass through the distance between two adjacent threads, it is necessary, as we have seen, that the power should make an entire revolution. This condition is unalterable, and there are many occasions on which an important use may be made of it. When it is proposed, for example, to measure the different parts of a very small space AB, Fig. 120.
it may be done by causing this space to be described by the extremity E of a screw DE, the threads of which are accurately formed at the same distance from each other throughout. If this screw be made to carry, at its other extremity, a dial GIH, the divisions of which, as the screw turns, pass under the fixed index FI; having ascertained what number of turns the screw must make in order that the point E shall describe the known extent AB, we shall be able, by the number of revolutions and parts of a revolution performed in causing the point E to pass over any part of AB, to determine the length of this part, however small it may be. If for instance the distance of the threads asunder be one tenth of an inch, and the circumference of the dial GIH five inches, any point of the circumference will move through five inches while the point E advances one tenth of an inch; consequently, the circumference being divided into tenths of an inch, while one division passes under the index, the point E would be carried forward only $\frac{1}{50}$ of $\frac{1}{10}$, or $\frac{1}{500}$ of an inch. This is called a *micrometer screw.*

220. By applying the screw to other machines their effect is greatly increased. If the power q, for example, applied to the
Fig.121. winch DEq, is made to turn the screw DC, the threads of which, acting upon the teeth of the wheel W, cause it to turn, and with it the cylinder I, around which passes the cord Kp carrying the weight p; the ratio of the power q to the weight p, may be determined thus. Calling q' the force exerted by the thread of the screw upon one of the teeth of the wheel W, we shall have

215. $$q : q' :: AB : circum.\ DE,$$

AB being the distance between two threads of the screw, and *circum.* DE denoting the circumference of the circle described
215. by the power q. The force q' is a power, which, applied to the circumference of the wheel, is exerted against the weight p; ac-
186. cordingly we have

$$q' : p :: IK : IL,$$

and, by taking the product of the corresponding terms of the two proportions,

$$q\,q' : q'p :: AB \times IK : circum.\ DE \times IL,$$

or,

$$q \quad : p \quad :: AB \times IK : circum.\ DE \times IL;$$

by which it will be seen that q has so much the more advantage according as AB and IK are smaller, considered with reference to *circum.* DE and IL. DC in this case is called a *perpetual screw.*

Of the Wedge.

Fig.122. 221. The *wedge* is a triangular prism intended to be introduced into a cleft for the purpose of enlarging it, or between two surfaces, in order to separate them further from each other, or to fix them at a determinate distance.

The action of the wedge, considered as an instrument for cleaving, is essentially modified by friction and other causes. As

there are no bodies which have not a certain degree of flexibility, the parts of the cleft in contact with the faces of the wedge may be separated further from each other without the extremity Z of the cleft shifting its place; so that a part of the force applied at the back DE of the wedge is employed in simply bending the two branches which form the cleft; and the other is exerted in distending the fibres of the part that has not yet yielded.

222. As this resistance depends upon causes so numerous and so variable at the same time, it is not to be expected that the nature and operation of the wedge, considered physically, will ever be reduced to a clear and satisfactory theory. In a mathematical point of view, the following explanation seems to be unexceptionable.

223. We suppose the direction of the power p, to be perpendicular to the back of the wedge, since if it is not, it may always be decomposed into two others, one perpendicular, and the other parallel to the back, of which the latter is incapable of urging the wedge backward or forward. This perpendicular force, therefore, may be considered as keeping the wedge ABC in equilibrium, while pressed at I, K, by the parts of a body that tend to unite. The theory of the inclined plane is accordingly applicable to this case, and the resistance exerted at I, K, cannot destroy the action of the power p, except while this power admits of being decomposed into two others q, r, passing through these points, and directed perpendicularly to the faces BC, AC, of the wedge. Therefore, the forces p, q, r, must meet in the same point E, be in the same plane ABC, and have the following proportion to each other, namely, Fig. 123. 45. 35. 211.

$$p : q : r :: \sin qEr : \sin pEr : \sin pEq,$$

or, since the sines of the angles qEr, pEr, pEq, are equal respectively to the sines of their supplements C, A, B, Geom. 80. Trig. 13.

$$p : q : r :: \sin C : \sin A : \sin B,$$
$$:: AB : BC : AC,$$

that is, the three forces p, q, r, are to each other as the three sides of the triangle to which their directions are perpendicular.

224. The three straight lines AB, BC, AC, are to each other as the faces of the wedge to which they respectively belong, for these faces are parallelograms of the same base, and whose alti-
Geom. 175. tudes are AB, BC, AC; it follows, therefore, that the power p and its two components, are to each other as the back and two sides of the wedge, or in other words, that *the power being represented by the back of the wedge, the force exerted against the two sides will be represented by these sides respectively.* A very acute wedge, therefore, or one whose sides are very long compared with the back, possesses an advantage in the same proportion, and may be made to exert a great power by means of a very moderate blow on the back.

225. There has not been a perfect agreement among mechanical writers as to the theory of the wedge. The direction of the resistance has sometimes not been sufficiently attended to, and the circumstance of one of the resistances proceeding from an immovable obstacle in certain cases, has sometimes been overlooked.

226. To the wedge are referred all cutting instruments, as knives, scissors, the teeth of animals, &c. A saw is a series of wedges on which the motion impressed is oblique to the resistance. A wimble is a combination of the screw and the wedge.

To the wedge of the pyramidal form, are reduced all piercing instruments, as nails, bayonets, stakes, piles, &c.

General Law of Equilibrium in Machines.

227. By combining together, in different ways, the machines above considered, we can form others, the number of which may be multiplied without end. With respect to compound machines, we determine the ratio of the power to the resistance necessary to an equilibrium, by having regard to the tensions of the cords that connect the different parts, this ratio being supposed to be known for each of the component parts.

But however complicated the machine, there is a simple rule by which the ratio of the power to the resistance is obtained directly

from the ratio of the spaces which the points of application of the two forces tend to describe in the same time. This is a particular case of what is called *the principle of virtual velocities.*

228. Let us suppose a very small motion given to the machine, and that the points of application of the two forces, describe curves to which the directions of these forces are tangents. Let u denote the velocity of the force p, or the space described in any given time by p, and v the corresponding velocity of q, or the space described by q in the same time; if the ratio of q to p, necessary to an equilibrium, is required, we shall obtain it very nearly by the proportion

$$q : p :: u : v;$$

and this will approach so much the more nearly to the exact ratio, according as the motion impressed upon the machine is less, so that by taking the limit of the ratio of u to v, we shall have exactly Trig. 16.
the ratio sought of q to p.

If the directions of the forces q, p, are not tangents to the curves described by the points of application of these forces, we take, instead of the spaces described by these points, the projections of these spaces upon the directions of the forces, and the inverse ratio of these projections will be the ratio of the forces, in the case of an equilibrium. We are at liberty to take for the point of application of each force, any point we may choose in its direction, provided we regard it as firmly attached to the machine.

229. We shall now apply this rule to a few examples in order to show its truth and utility.

In the lever, I take for the points of application of the forces p, q, the feet L, M, of the perpendiculars FL, FM, let fall from Fig. 124.
the fulcrum F, upon the directions of the forces. When the lever turns about the point F, the points L, M, will describe the similar arcs LL', MM', to which the directions $p\,L$, $q\,M$, of the forces, are tangents. The lengths of these arcs are to each other as their radii FL, FM; so that we have the proportion, Geom. 238.

$$LL' : MM' :: FL : FM,$$

that is,

$$u : v :: FL : FM;$$

and as this ratio remains the same, however small the motion impressed upon the lever, it holds true, when $u : v :: q : p$, which gives exactly

$$q : p :: FL : FM.$$

In the case of an equilibrium, therefore, the two forces are to each other in the inverse ratio of the perpendiculars let fall upon their directions, as already determined by a different method.

230. If we had taken the extremities B, D, of the lever for the points of application of the forces p, q, the directions of the forces would no longer be tangents to the arcs described by these points. We should therefore have to project these arcs upon the straight lines Bp, Dq, then to take the ratio of these projections, and seek the limit of this ratio.

On the supposition of motion, the angles DFD', BFB', described by the arms FB, FD, are equal; and the arcs BB', DD', described by B, D, about the point F, as a centre, are to each other as the radii FB, FD. This ratio continues the same when the arcs become infinitely small, so that we have constantly

$$FB : BB' :: FD : DD'.$$

From the points B', D', let fall the perpendiculars $B'A$, $D'C$, upon the directions of the forces p, q; and we shall have

$$u = BA, \qquad \text{and} \qquad v = DC.$$

Also from F, let fall the perpendiculars FM, FL, upon the directions of the forces. By considering the infinitely small arcs
Geom. 209. BB', DD', as straight lines, perpendicular respectively to the radii FB, FD, the triangles $DD'C$, FMD, are similar, as also the triangles $BB'A$, FLB; whence

$$FB : FL :: BB' : BA = \frac{BB'}{FB} \times FL,$$

and

$$FD : FM :: DD' : DC = \frac{DD'}{FD} \times FM;$$

accordingly we have, by substitution,

$$u = \frac{BB'}{FB} \times FL, \text{ and } v = \frac{DD'}{FD} \times FM,$$

and hence

$$u : v :: \frac{BB'}{FB} \times FL : \frac{DD'}{FD} \times FM,$$

that is, since Geom. 288.

$$\frac{BB'}{FB} = \frac{DD'}{FD},$$

$$u : v :: FL : FM;$$

from which we obtain as before,

$$q : p :: FL : FM.$$

231. In the wheel and axle, when motion commences, the Fig. 102. points of application of the power and resistance describe similar arcs, or arcs of the same number of degrees, the one upon the circumference of the wheel, and the other upon that of the axle. The directions of the forces are tangents to these arcs, whose lengths are to each other as the radii IB, IA, of the axle and wheel; in this machine, therefore, we have

$$u : v :: IB : IA,$$

and accordingly

$$q : p :: IB : IA.$$

232. In the pulley, if, while the weight p describes in rising a space equal to u, each of the two cords which meet at the mova- Fig. 95. ble pulley, is shortened by the same quantity u, the cord to which is suspended the power q, will be lengthened by a quantity equal to $2\,u$, which will consequently be the space passed through by q in its descent. Taking, therefore, $v = 2\,u$, we shall have, in the case of an equilibrium,

$$\begin{aligned} q : p &:: u : v, \\ &:: 1 : 2; \end{aligned}$$

or generally, the cords being parallel,

$$q : p :: 1 : n,$$

n denoting the number of cords that meet at the movable pulley.

I will take, as the last example, the assemblage of pulleys, represented in figure 87. If the power q, in descending, pass over a space v, the point N will be elevated by a quantity

equal to $\frac{1}{2}v$; calling this v', the point M will be elevated by a quantity equal to $\frac{1}{2}v'$ or $\frac{1}{4}v$; this being designated by v'', the point L will be elevated by a quantity $\frac{1}{2}v''$ or $\frac{1}{8}v$; and this is the space through which the weight p passes in rising; calling this u, therefore, we shall have, whatever v may be, $u = \frac{1}{8}v$, which gives in the case of an equilibrium

$$q : p :: u : v,$$
$$:: 1 : 8;$$

or generally

$$q : p :: 1 : 2^n,$$

n denoting the number of movable pulleys in the system, which agrees with what has been before shown.

233. In the screw, when the nut passes over a space equal to the distance between two adjacent threads, or when it is elevated

Fig. 118. through a height equal to DE, the point to which the power is applied describes a spiral about the axis AB, and rises through a space equal to DE, the projection of which spiral upon a horizontal plane is a circle, of which BG is the radius. Moreover, these motions of the nut, of the point of application, and its projection are such that if the nut describes one half, one third, or any other part of the distance between two threads, the point of application will describe a similar part of the length of the spiral, and its projection a similar part of a circumference of which BG is the radius; accordingly, if we call u the height through which the nut rises, and v the arc of a circle described in the same time by the horizontal projection in question, we shall have

$$u : v :: DE : \textit{circum. } BG.$$

Now the direction of the force q not being a tangent to the spiral, it is necessary, in order to apply to this case the general law of equilibrium, to consider the projection of a very small arc of the spiral upon the direction of the force q; but this direction being supposed to be a tangent to the circumference above mentioned, the projection upon this tangent will be very nearly equal to the projection upon the circumference, and one may be taken for the other, when we consider only an infinitely small motion of the nut; consequently, the ratio of the force q, to that of p, will be given, by

taking the limit of the ratio of u to v. Now as this ratio is constantly the same, and equal to the ratio of DE to *circum.* BG; according to the preceding proportion, we shall have, in the case of an equilibrium,

$$q : p :: DE : circum.\ BG,$$

as before shown by a different process.

Of Friction.

234. The surfaces of bodies, even the most smooth, are covered with elevations and depressions; and when two bodies are brought in contact with each other, the prominent parts of the one enter the cavities of the other, and they cannot be moved the one over the other, without employing a certain force. The resistance arising from this cause is called *friction.*

There are two sorts of friction, one which takes place when the bodies in contact have simply a sliding motion; the other when one or both the bodies move by turning upon an axis. We have an instance of the former in the motion of skates and sledges, and of the latter in the action that exists between the wheels of wheel-carriages and the ground. The resistance arising from the second kind of friction, is much less than that of the first, since a rotatory motion serves to disengage the parts in contact without breaking down the eminences or lifting them out of the cavities.

When the rising up of one body over the other is completely prevented, the friction becomes very intense, as appears in what may be accounted an extreme case, the drawing of wire, and the rolling of bars of iron, copper, &c., into plates, where the extension of the metal is the effect of the friction acting all round or on two sides.

If the asperities with which the surfaces of bodies are covered, were perfectly hard, and immovably attached to these surfaces, it would be necessary, in overcoming the friction, to raise the incumbent mass. If these asperities, on the other hand, were perfectly flexible, there would be no resistance and no friction.

But as neither supposition is true in any case, it follows, (1.) That the resistance of friction arises in part from the difficulty of bending the asperities, and in part from the necessity of raising in a degree the body or incumbent mass; (2.) That, the asperities having only a limited degree of adhesion, when the force necessary to cause the body to move exceeds this, the asperities yield, or are broken down, and the surfaces are gradually worn. Thus the effect of friction in machines is not only to consume a part of the force employed, but also to destroy the machines themselves.

It would seem difficult, if not impossible, to establish general rules, sufficiently exact for determining the force of friction. Indeed it will be readily seen, that this resistance must vary according to the nature and texture of the surfaces in contact, their flexibility, and the adaptation in size and figure of the prominent parts and cavities to each other, and according as the force is greater or less by which the surfaces are pressed together; moreover, on account of the flexible nature of surfaces, the prominent parts are found to penetrate to a greater depth when more time is allowed for enlarging the openings which they tend to enter.

It belongs to experiment alone to enlighten us upon these points, and to teach us the proportional effect due to each. The information, however, derived from this source, is not yet so perfect and complete as could be wished, though it is such as may be useful on many occasions. We proceed now to make known some of the results of experiments, as also the method of applying them in calculating the effect of friction in the different kinds of machines, and the different kinds of motion.

235. (1.) When the surfaces which are to rub the one upon the other, are of the same kind of matter, the resistance of friction, other things being the same, is greater than when the surfaces are of different kinds. Thus, two pieces of wood of different kinds slide upon each other with less difficulty than two of the same kind. Iron rubbing on copper has less friction than iron on iron or copper on copper. This is supposed to be owing to the prominent parts and cavities being more nearly fitted to each other in the latter case than in the former.

(2.) The more rough the surfaces, or the less they are planed or polished, the greater is the friction. This resistance, therefore, may be diminished, by smoothing the surfaces, or by filling the openings and pores with other matter, as oil, soap, grease, black-lead, &c., with any substance indeed, which, while it fills the cavities, does not give rise to a new adhesion.

(3.) It would seem that the extent of surface ought to contribute sensibly to the friction; it appears, however, by a great variety of experiments, that this circumstance makes but little difference; we find in fact the same difficulty for the most part in drawing a body upon one of its surfaces as upon another, though very different in extent, provided they are equally smoothed. Thus, oak rubbing on oak, is found to have a friction of about 44 per cent.; and on diminishing the surface as much as possible, this is reduced only to $41\frac{1}{2}$ per cent. We must except, however, the case of bodies resting upon a point, when the friction is more considerable, than when the contact takes place in several points.

(4.) It is principally from pressure that friction arises, and this resistance is found to increase in proportion to the pressure; that is, we require twice the force to overcome the friction when the weight is doubled, other things being the same.

(5.) Still the time, during which the two surfaces are acting upon each other, either by their own gravity, or by any other force, is to be taken into the account, although its effect has not yet been accurately determined; it is found that the augmentation depending on this cause, has its limits, and that these limits vary according to the nature of the rubbing surfaces. Coulomb found that in wood sliding on wood, without grease, the friction at first increased, but in a minute or two came to a limit, which it did not afterwards exceed. Oak, for example, sliding on oak, though the pressure was varied from $74^{\text{lb.}}$ to $2474^{\text{lb.}}$ had a friction after a minute always nearly 44 parts in the hundred.

In the case, however, of iron rubbing on iron, or iron on brass, the friction is the same whether the bodies are just beginning to move from rest, or have acquired any given velocity.

When heterogeneous bodies are made to slide upon one another, as wood on metal, the friction increases slowly with the time, and does not arrive at its maximum in less than four or five days. Iron on oak after ten seconds, is found to have a friction of $7\frac{4}{5}$ per cent., and after four days it amounts to nearly 20 per cent. In heterogeneous substances, too, the friction increases sensibly with the velocity, and follows nearly an arithmetical, while the velocity follows a geometrical progression.

When the surfaces are smeared with some unctuous substance, although the friction is diminished, a certain time is required in order that the friction may attain its maximum. Oak rubbing on oak, the surfaces being covered with tallow, has a friction that continues to increase for five or six days, and becomes stationary at about 42 or 43 per cent. In the case of brass on iron with fresh tallow between the surfaces, the friction is four days in coming to a maximum, when it is 10 or 12 per cent., it being 9 per cent. at the commencement. The increase where metals are used, is much less considerable than in experiments with substances more porous and yielding.

236. The quantity of friction being determined for a particular kind of matter, let us now see if the effect upon a given machine, or given motion, may be thence deduced, friction being considered as simply proportional to the pressure.

Fig. 126. Let us take, as the first example, the body p, situated upon a horizontal plane AB, and drawn by the weight of the body q, parallel to AB. Suppose that the body q has a weight just sufficient to put the body p in motion. The ratio of the weight q to the friction is thus found.

From the centre of gravity G of the body p, let fall the perpendicular GH upon the plane AB. The body p is urged by gravity in the direction GH, and by the weight q in the direction KM which meets GH in K. From the joint action of these two forces, there will result an effort according to some line KI, meeting in I, the horizontal plane AB; and this effort must be just counterbalanced, since we have supposed that the body p is only upon the point of moving. Suppose the effort according to KI or KIZ applied at the

point I, and decomposed into two others, the one perpendicular to the plane AB, and the other in the direction of this plane. These efforts will evidently be the same as those which were directed according to KH and KL. Moreover the first will be destroyed, especially if it meet the plane AB in some point I common to this plane and the surface of the body. As to the second, since it is in the direction of friction, it will not be destroyed unless it happen to be exactly equal to the force of friction.

We hence see how the value of friction is to be determined; we take successively for q different weights until we find one that is just sufficient to cause a motion in the body p. But not to comprehend, in estimating the friction of the body p, effects foreign to that which is sought, it is necessary to attend to several particulars; (1.) The pulley M should move with the greatest ease, and the cord $KM\ q$ should be as flexible as it can be made. (2.) The cord CM should be attached to some point C as near as possible to the plane AB. The necessity of this precaution arises from the circumstance, that other things being the same, the point I, where the effort in the direction KI meets the plane AB, will approach so much the nearer to the extremity E of the base of the body, or will fall without the base so much the farther from the extremity E, according as the point C is the more elevated above the plane AB. Now in the case where the point I falls without the base, the effort perpendicular to the plane not being entirely destroyed, there will hence result a tendency in the body to rotate; and the friction thence arising would be somewhat more considerable than the proper friction in question.

237. Let us now consider a weight p, put upon an inclined plane, Fig. 127.
and retained by the effect of friction alone. The action of gravity directed according to the vertical GZ passing through the centre of gravity G of the body, and meeting in I some point of the plane AB, may be decomposed into two parts, one in the direction of the plane, and the other perpendicular to it. The second will be destroyed, if the point I does not fall without the base DE, and the first in order to be destroyed must be equal to the force of friction. Now it is evident that by forming the parallelogram $IHZL$, IZ will represent the weight of the body, IH the pressure, and HZ or

IL the force of friction; hence the triangles *IHZ*, *ABC* being
Geom. 202. similar, we have

$$HZ \quad \text{or} \quad IL : IH :: BC : CA,$$

from which it will be seen, that the force of friction is to the pressure as the height of the plane to its base.

It will be perceived, in like manner, that

$$HZ : IZ :: BC : AB;$$

that is, the force of friction is to the weight of the body as the height of the plane is to its length.

In order, therefore, to determine the friction in different substances, we have only to raise the plane *AB* till the body *p* is upon the point of moving; then measuring the height and the base, we shall have the ratio of the force of friction to the pressure.

238. By these two examples, it will be seen, that regard being had to friction, the condition required in order that a body may remain in equilibrium upon a proposed surface, and be in a state approaching the nearest to motion, is, that the single force which acts upon it, if there be but one, or the resultant of all the forces, if there be several, have with respect to the surface upon which it is to move, an inclination *GIE* or *LIZ*, such that *IL* shall be to *HI*, as the force of friction is to the pressure. But *IL* is to *LZ*, as one
Trig. 30. is to the *tang*. *LIZ*, 1 being the radius of the tables. Consequently the inclination *LIZ* must be such that the radius shall be to the tangent of this inclination, as the force of friction is to the pressure; therefore, the ratio of the force of friction to the pressure being once ascertained, it will always be easy to determine the inclination belonging to the resultant of all the forces which act upon the body, this body being in a state of equilibrium approaching as near as possible to motion. Hereafter we shall call the angle *LIZ* *the angle of friction*. It is different for different substances, and for different degrees of smoothness of the same substance. If the friction is $33\frac{1}{3}$ per cent., or one third of the pressure, which is nearly the case with respect to a great many kinds of matter, the rubbing surfaces being tolerably smooth, the tangent of *LIZ* will be triple of the radius, that is, *LZ* will be equal to 3 *IL*. Now the angle

of which the tangent is triple the radius is 71° 34′. This will accordingly be the angle of friction in these cases.

239. By means of this result, it will be easy to determine in each machine what ratio ought to exist between the power and the weight in order that motion may be upon the point of taking place, allowance being made for friction.

In the lever, for example, let us suppose that the fulcrum is a simple support, as represented in figure 128. We have seen, that with respect to this machine, there cannot be an equilibrium, unless the resultant DF of the two forces p, q, be perpendicular at F to the common tangent to the surface of the lever and that of the fulcrum. On the supposition of friction, the case is different; it 163.
is still necessary that the resultant should be directed from the point D to the fulcrum F; but it is sufficient in order to an equilibrium, that one of the two inclinations DFA, DFB, according as we wish q or p to prevail, should be greater than the angle of friction, and for the state of equilibrium approaching the nearest to motion on the part of the power q, it suffices that the inclination DFA should be precisely equal to the angle of friction; since, if we imagine the force according to DF decomposed into two others, the one perpendicular to AB, and the other in the direction AF, the force in the direction AF will be less than the friction in the first case, and exactly equal to it in the second. With respect to the two forces q, p, they will still be in the inverse ratio of the perpendiculars FK, FL.

240. But if the fulcrum is such that the lever can have no other motion except that of a rotation, that is, if it turn on an axis or pin, we adopt the following method which is common to the lever, the pulley, and the wheel and axle, especially when in this last machine the power and weight are in the same plane. We shall consider first the wheel and axle; the manner of treating the lever and pulley will afterwards appear.

Let HCI be the plane of the wheel, GKL a section of the Fig.129.
cylinder, and NDM the axis about which the machine is to turn. On the supposition that there is no friction, the resultant of the two powers p, q, passing through their point of meeting A, must pass also through the centre F of the axis. But in the case of

friction, the machine must remain in equilibrium so long as the direction of the resultant, supposed to be AD, does not make with the surface NDM (that is, with the tangent at the point where AD meets this surface,) an angle less than the angle of friction. This will be evident by imagining the force in question decomposed into two others, one perpendicular to the tangent at D; and the other in the direction of this tangent.

This being done, since AD is the directon of the resultant, we
48. shall have

$$q : p :: \sin\ GAD : \sin\ DAC,$$
$$:: \sin\ (GAF + FAD) : \sin\ (FAC - FAD).$$

Now, (1.) If we let fall upon AD the perpendicular FE, in the right-angled triangle FED, the angle FDE is the complement of the angle EDO which AD makes at D with the surface NDM, and is accordingly supposed to be known; so that if we call f the angle of friction, FDE will be the complement of f, and if we call δ the distance FD, or the radius of the axle, we shall have

Trig. 30. $$FE = \delta \cos f,$$

the radius of the tables being equal to 1. (2.) As the directions of p and q are supposed to be known, as well as the dimensions of the machine, the angles GAF, FAC, are supposed to be known as also the distance AF. Thus, in the right-angled triangle FAE in which AF, FE, are known, it will be easy to calculate the angle FAE;
Trig. 30. calling this angle e, and the angles GAF, FAC, a, b, respectively,
48. we shall have

$$q : p :: \sin\ (a + e) : \sin\ (b - e);$$

consequently $q = \dfrac{p \sin\ (a + e)}{\sin\ (b - e)}$; this is the value of the power in the case of friction.

If the friction is nothing, the angle f of the friction is 90°; that is, the resultant must be perpendicular to the surface of the axis, and consequently pass through the centre F. We have accordingly, $\cos f = 0$, and hence $FE = 0$, and $e = 0$; therefore

$$q = \frac{p \sin a}{\sin b}.$$

Now regarding FA as radius, the perpendiculars FG, FC, are the sines of the angles FAG, FAC, or of a, b; we have, therefore, by calling FG, D, and FC, D',

$$D : D' :: \sin a : \sin b,$$

whence

$$\frac{\sin a}{\sin b} = \frac{D}{D'}$$

and

$$q = \frac{p\,D}{D'}, \text{ or } q : p :: D : D',$$

which agrees with what has already been proved. 178.

If the directions $p\,A$, $q\,A$, are parallel, the angles GAF, FAE, FAC, are considered as infinitely small, and consequently as having the same ratio as their sines. We may accordingly sub- Trig. 17.
stitute $\sin a + \sin e$ for $\sin(a + e)$, and $\sin b - \sin e$ for

$$\sin(b - e),$$

and we shall have

$$q = \frac{p(\sin a + \sin e)}{\sin b - \sin e}.$$

But we have just seen, that

$$D : D' :: \sin a : \sin b,$$

and for the same reason,

$$\sin a : \sin e :: FG : FE,$$
$$:: D : \delta \cos f;$$

whence

$$\sin a = \frac{D \sin b}{D'},$$

and

$$\sin e = \frac{\delta \cos f \sin a}{D} = \frac{\delta \cos f \sin b}{D'};$$

substituting in the value of q, for $\sin a$ and $\sin e$ the values above found, we have

$$q = \frac{p\left(\frac{D \sin b}{D'} + \frac{\delta \cos f \sin b}{D'}\right)}{\sin b - \frac{\delta \cos f \sin b}{D'}} = \frac{p\,(D + \delta \cos f)}{D' - \delta \cos f}.$$

Therefore, since when there is no friction, the value of the power is $\frac{p\,D}{D'}$, if we call z the augmentation which the power must receive on account of friction, we shall have

$$z = \frac{p\,(D + \delta \cos f)}{D' - \delta \cos f} - \frac{p\,D}{D'},$$

$$= \frac{p\,D'\,(D + \delta \cos f) - p\,D D' + p\,D\,\delta \cos f}{D'\,(D' - \delta \cos f)},$$

$$= \frac{p\,(D' + D)\,\delta \cos f}{D\,(D' - \delta \cos f)};$$

from which it will be seen, that the effect of friction will be less according as the radius of the axle is less, although it does not diminish exactly in proportion to this radius.

241. This solution adapts itself to the lever, by regarding D' and D as the distances of the directions of the two forces from the fulcrum. It is applicable also to the fixed pulley, by supposing $D' = D$, which would give,

$$z = \frac{2\,p\,\delta \cos f}{D - \delta \cos f}.$$

Although in the wheel and axle we have supposed the directions in the same plane, the solution is not the less adapted to the ordinary construction of this machine in which the weight and power are exerted in planes but little distant from each other.

242. To ascertain the effect of friction in the movable pulley
Fig.130. we proceed thus. In order that the power q may be upon the point of causing the pulley to turn about its axis F, it is necessary that it should receive an augmentation sufficient to overcome the friction. Now by this augmentation, the power causes the weight p to depart from its position to such a degree, that a vertical drawn

through the centre of gravity shall make at the point D, with the surface of the axle, an angle equal to that of friction. Then any increase of the power will cause the pulley to turn.

By the position which the weight has taken, it will tend to turn the axle upon its centre F, with a force, the moment of which will be $p \times FE$, FE being perpendicular to a vertical passing through D. Now we have already seen that

$$FE = \delta \cos f;$$ 240.

this moment, therefore, will be $p\,\delta \cos f$. But in order that the power, by its augmentation, which I shall call z', may be upon the point of overcoming this effort, it is necessary that the moment $z' \times FG$ of the force z', with which it tends to cause a rotation about F, should be equal to the moment $p\,\delta \cos f$; we have, there- 172.
fore, by calling the radius FG, D,

$$D\,z' = p\,\delta \cos f,$$

and consequently,

$$z' = \frac{p\,\delta \cos f}{D};$$

from which it will be seen, that the effect of friction will be less according as the radius of the axle is less than that of the pulley, and in the same ratio.

It will hence be easy to determine the effect of friction in the different systems and combinations of pulleys. Suppose that the question related to one similar to that represented in figure 92, the weight p being equal to $400^{\text{lb.}}$, and the radius δ of the axle as well for the fixed as for the movable pulleys being a fifth part of the radius D of the pulley.

The two cords 1, 2, sustain half of the weight or $200^{\text{lb.}}$; thus, if in the value of z, we put for p $200^{\text{lb.}}$, $\frac{1}{5} D$ for δ, and on the supposition that the friction is one fourth of the pressure, which would give the angle of friction equal to $75° 58'$, and the $\cos f$ or $\cos 75° 58$, equal to 0,24249,* or simply 0,24, we shall have,

* See table of natural sines and cosines, *Topography*.

$$z' = \frac{p\,\delta \cos f}{D},$$

$$= 200 \times \tfrac{1}{5} \times 0{,}24 = 9^{\text{lb.}}, 6,$$

which is an expression for the excess of tension of the cord 2 over the cord 1, on account of friction.

Now the cord 2 and the cord 3 passing over the fixed block, we shall have the excess of the tension of the cord 3, over the
241. cord 2, by the formula,

$$z = \frac{2\,p\,\delta \cos f}{D - \delta \cos f},$$

in which p represents the tension of the cord 2, or $109^{\text{lb.}}, 6$; since, without friction, this tension would be one fourth of. the weight or $100^{\text{lb.}}$. Accordingly we have

$$z = \frac{219{,}2 \times \frac{1}{5} \times 0{,}24}{1 - 0{,}24 \times \frac{1}{5}} = \frac{10{,}522}{0{,}95} = 11^{\text{lb.}}, 1.$$

This is the quantity by which the cord 3 is more stretched than the cord 2, on account of friction. Thus the cord 3 is stretched with a force equal to $120^{\text{lb.}}$ 7.

As the cords 3 and 4 embrace the movable block, we determine how much the cord 4 is more stretched than the cord 3 by the formula

$$z' = \frac{p\,\delta \cos f}{D},$$

which would give as above $z' = 9^{\text{lb.}}, 6$. Thus the cord 4 is stretched with a force equal to

$$100 + 9{,}6 + 11{,}1 + 9{,}6 = 130^{\text{lb.}}, 3.$$

Finally, by supposing for greater simplicity, which will make but little difference in the result, that the two cords 4, 5, which embrace the fixed block, are parallel, we shall have the quantity by which the cord 5 is more stretched than the cord 4 by the formula

$$z = \frac{2\,p\,\delta \cos f}{D - \delta \cos f},$$

which gives

$$z = \frac{260,6 \times \frac{1}{5} \times 0,24}{1 - \frac{1}{5} \times 0,24} = 13^{\text{lb.}}, 2.$$

Thus the tension of the cord 5, which without friction would be only 100, is equal to $130,3 + 13,2 = 143^{\text{lb.}}, 5$.

243. In the determination, which we have given, of the effect of friction upon the movable pulley, we have not allowed for any increased pressure at D arising from the augmentation of the power, although this is done by some writers. The reason Fig. 130. is, that this augmentation of the power contributes nothing to the pressure at D; indeed, setting aside the stiffness of the cords, and certain other obstacles, whenever the power is greater than is necessary to an equilibrium, the body of the pulley is elevated by this excess. The augmentation of the power does not contribute any thing to the friction against the axis, if no regard is had to the velocity that may thus be produced, which together with the inertia of the weight and pulley are at present left out of consideration. It is only the weight which presses. This is not the case with the fixed pulley; and the method which we have given comprehends, with respect to this point, every thing which ought to be taken into the account, although it differs from the course heretofore adopted, in which something has been taken for granted.

We will not deny, that in the calculation which we have given of the effect of friction, if we would know the effect of friction very rigorously, it would be necessary to consider the subject a Fig. 92. little differently. In fact, by determining in the way we have done, the particular tensions of each of the cords, it is taken for granted that each pair of cords acts as it would do in the case of a simple pulley, which is not strictly true, perhaps. But this approximation will suffice for the present.

244. In order to determine upon the inclined plane, the ratio of the power to the weight, when the former is upon the point of causing the body to move, we proceed thus. Through the point of meeting F, of the directions of the power q, and weight p, we Fig. 131. suppose the line FI drawn, making with the plane AB an angle FIA equal to that of friction. In order that the power q may be upon the point of causing motion, it is necessary, (1.) That the

resultant of the power and weight should be directed according to FI. (2.) That the point I, where the line FI meets the plane, should belong to some point of the base DE; otherwise the body would tend to turn.

This being premised, we have

$$p : q :: \sin q\,FI : \sin p\,FI,$$

or, letting fall upon the plane the perpendicular FH,

$$p : q :: \sin (q\,FH - HFI) : \sin (p\,FH + HFI).$$

Now the angle HFI is the complement of the angle of friction; and the angles $q\,FH$, $p\,FH$, are supposed to be known, since the direction of the power is considered as known, together with the
Geom. 209. inclination of the plane, which is equal to the angle $p\,FH$; we have accordingly the ratio of p to q.

If we would determine this ratio in lines, we have only to draw through any point B of the inclined plane, the line BT, making with AB the angle $ABT = HF\,q$, and the line BV, making with AB the angle $ABV = HFI$, the complement of the angle of friction. Then drawing the horizontal line AT, we shall have

$$p : q :: VT : BT,$$

Geom. 78. since the angle $VBT = ABT - ABV = HF\,q - HFI$, and the angle $BVT = BAV + ABV = p\,FH + HFI$. Now in the triangle BVT

Trig. 32. $$VT : BT :: \sin VBT : \sin BVT.$$

Instead of making the angle $ABT = HF\,q$, and the angle $ABV = HFI$, we may draw BT perpendicular to the direction of the power q, and BV perpendicular to FI; this amounts to the same thing, and is moreover analogous to the course pursued in article 203.

245. The second condition shown to be necessary in order that the power q may be upon the point of moving the body, renders it evident, that when the body does not rest upon a point, the direction of the power produced must meet the vertical,
Fig. 132. drawn through the centre of gravity, at the point F, where this last line is met by the line IF, proceeding from some point of con-

tact I, and making with the plane an angle equal to the angle of friction.

246. We proceed in the same manner in determining the second kind of friction, or that which is to be overcome in giving a rolling motion to bodies terminated by curved surfaces; I say curved surfaces, since with respect to bodies terminated by plane surfaces, as they cannot roll, except by turning on a point or some angular part, we shall not treat of these, the laws and value of friction in such cases not being sufficiently known. But as to the friction of bodies terminated by curved surfaces, the method is precisely the same as that above pursued. We have only to suppose the angle of friction to approach more nearly to 90°; and it is to experiment that we are to look for the determination of this angle in all cases.

247. Friction may be the occasion of motions very different from those which take place without this cause, some of which it may be worth while to notice.

We have already mentioned more than once, what must hap- 136, &c.
pen to a free body BOq, which receives an impulse in a direc- Fig. 133.
tion not passing through the centre of gravity. But if the body were struck externally according to any direction AB, it would not receive the whole of this impulse. The impelling force is to be decomposed into two others, one in the direction of a tangent to the surface, and the other perpendicular to this surface. When there is no friction the impelling force would have no effect in the direction of a tangent. It is only the force in the direction BF, therefore, which would be transmitted to the body, and this would not cause the body to turn, except when it happened not to pass through the centre of gravity G. It will hence be seen that if the body were of a spherical form and homogeneous, it would never be made to turn in virtue of an external force unaccompanied with friction, since in this case a perpendicular to the surface would always pass through the centre of the figure, which would, at the same time, be the centre of gravity. On the supposition of friction, the case is different. The force in the direction of a tangent would transmit itself by means of the asperities of the surface, and to a greater or less degree according to the amount of friction; so that in addition to the motion aris-

ing from the perpendicular force BF, the body would turn, and the centre of gravity G would advance in a line parallel to the tangent, as if the point B were drawn in that direction by means of a thread attached at this point, and with a power equal to the force of friction.

Fig. 134. 248. Let us suppose that a hard, spherical body ABC, falls freely upon a horizontal plane HR, and that it receives, from some cause or other, a motion of rotation about its centre of gravity; if there were no friction, this body, after meeting the plane would preserve only its rotatory motion, and its centre of gravity would be at rest. But in the case of friction, when the body has reached the plane, it will roll from I toward R, or from I toward H, according as the rotatory motion is in the direction CAB, or in that of BAC; since the resistance of friction, which is exerted in the direction of the plane, is equivalent to a force acting upon this body in a direction opposite to its motion; and as it does not pass through the centre of gravity of the body, it must give it a motion parallel to the plane, and a rotatory motion, both in the direction contrary
136. to the actual motion of rotation. Now of these two motions, the latter diminishes continually the original motion; and on the other hand the motion of the centre will be accelerated to a certain point, after which it will be diminished till it is destroyed with the motion of rotation.

249. We are hence enabled to explain several phenomena;
Fig. 135. as (1.) Why a spherical body ABC, struck in the direction DB, after having advanced in the direction IE, returns afterward from E toward I, and may even pass beyond I toward H. The impulse in the direction DB causes it to turn (on account of friction at B), according to ABC, and to advance in the line IE; but the friction upon the plane, being now a friction of the first kind, the motion of the centre of gravity is soon destroyed, and the motion of rotation gives rise to another in the opposite direction, as in the
248. preceding case.

(2.) We are moreover furnished, upon the same principles, with the reason why a cannon-ball, which had apparently lost nearly all its force, seems on striking to recover it again, and often with violence. When it is impelled by the force of the

powder, it acquires at the same time, in consequence of the friction against the bottom of the bore, a rotatory motion, which is but little affected while in the air; but when the ball comes to reach the ground, as the rotatory motion on the part toward the ground takes place in a direction opposite to the progressive motion, the consequence must be an acceleration in the motion of the centre, that is, of the progressive motion. 248.

250. Finally, if friction is disadvantageous in many cases, it is still not without its utility. Were it not for this, the least inclination would be continually subjecting us to a fall. No man or other animal could turn while in rapid motion without falling, Fig. 136. whatever position he might take; whereas, on account of friction, an animal may incline himself toward the point *F*, for example, about which he is moving, in such a manner that his weight, directed according to the vertical *GK*, passing through the centre of gravity *G*, and the tendency to fly off *GC*, acquired by turning, and which is directed from *F* toward *C*, will conspire to produce a single force according to the line *GI*, passing through a point *I* between the legs of the animal; this force, although oblique, is still destroyed by friction, provided the inclination be within the limits required by the laws of friction.

It is moreover to friction that we are indebted for the power of diminishing friction, when injurious; since it is only by means of this resistance that we are able to work and polish the surfaces of bodies. It is to friction that we owe the facility with which the parts of certain machines are rendered sometimes fixed, and sometimes movable. It is by friction that scissors and the like instruments, pincers, forceps, files, &c., produce their effect. If the parts of scissors, for example, were not saws, armed with small teeth, which take into the cavities of the bodies to be cut, these bodies would slip from between the two edges.

Friction is also very often of service in moving bodies in certain directions; thus if we would raise the body *p* by means of the lever *AB*, it is very easily done, by making the body bear on the edge *CD*. The friction in a case like this, being very Fig. 137. considerable, renders *CD* fixed, and prevents the body from slipping. The same cause keeps the extremity *A* of the lever in its

place. In this case, if we would know the ratio of the weight p to the power q, we imagine the weight of p, directed according to the vertical GK, passing through the centre of gravity G, decomposed into two parallel forces, the one passing through the point I in which the body rests upon the lever, and the other through a point of CD, situated in the plane of the two parallels
52. GK, IM; then the resulting force in I will be to p as EK to EM; and if from A we let fall upon IM the perpendicular AL, the force q will be to the force at I, as AL to AB; whence

$$q : p :: AL \times EK : AB \times EM.$$

In short, it is only on account of the friction at the point I that we regard the force according to IM, as transmitted entirely to the lever. The lever would otherwise receive only a part of this force which would exert itself in the direction of a perpendicular to AB.

251. It is to friction and friction only, that we are to refer the singular motion by which certain bodies in a state of rotation are seen to elevate themselves contrary to the tendency of gravity; I speak of the phenomena of the *top*. It is well known that when a body of this description, that is, one which is symmetri-
Fig. 138. cal with respect to one of its axes, as ND, has received a rotatory motion about this axis, and moves upon its point N, over a horizontal plane XZ; it is well known, I say, that the smaller the point N, and the greater the divergency of the sides from it, the greater is the tendency of this body to rise, and thus to place the axis ND in a vertical position. We proceed now to show that this phenomenon could not take place without friction; we shall speak, moreover, of the nature of this friction.

Fig. 139. To simplify the subject, let us consider only the axis ND of the top, and let us suppose the point N, and the horizontal plane XZ, to be perfectly smooth. The only cause which opposes the motion of the centre of gravity G, being the plane XZ, the resistance which the centre of gravity meets with can have no other direction than the line NK, perpendicular to XZ, whatever be the rotatory motion about the axis ND. Now it is evident that this resistance takes place only because gravity urges the body

towards the plane, for the rotatory motion about the axis ND cannot cause any pressure upon this plane; the resistance in question, therefore, can never be equivalent to the force of gravity, and there will always remain in the centre of gravity G, a force tending to bring it to the plane. Hence, when there is no friction, and the top has received at first no motion of rotation, except about the axis of its figure, it must of necessity fall.

252. It is not the same on the supposition of friction. In this case, the resistance which takes place at N, acts not according to the perpendicular NK, but according to the line NK', which makes with the plane XZ an angle equal to the angle of friction, and passes through N, one of the points of friction. Whatever may be this angle and this point, the resistance which takes place along the line NK', is equivalent to a force acting upon the body in a contrary direction; now, as this direction does not pass through the centre of gravity, it must produce in the body a rotatory motion, that is, a variation in its actual motion of ro- 137.
tation; but it must also transmit itself entirely to the centre of gravity. Let us suppose, therefore, that GL parallel to NK' is this force; if the vertical line GI represent the force of gravity, and the parallelogram $GLEI$ be completed, GE will be the actual force which belongs to the centre of gravity G.

Now, the angle LGI and the force GI remaining the same, the greater the force acting in the direction $K'N$, and consequently the greater the force GL, the more nearly will the line GE approach the line GL; that is, the greater will be the tendency of the point E to rise above G. It remains therefore to be seen, whether from the nature of friction, together with the figure of the body, and its motion of rotation, the ratio of the force in the direction NK' (or the force GL) to the force of gravity GI, can Fig. 138.
be increased till the point E shall be above the point G; in which case it is clear that the centre of gravity may rise with respect to the plane; yet not so as to cause the point N to quit it, because the motion of rotation which results from the force in the direction $K'N$, will tend to bring this point toward the plane. Now (1.) As the body rests upon a point, it cannot be denied that the parts of this point sink more deeply than if the body rested upon a sensible surface. Regard being had to the rotatory motion about ND, and to the action of gravity, the pres-

sure exerted upon N is by no means the effect of gravity only. To have a just idea of this pressure, we must consider that by gravity the parts of the point N are at first urged against the plane; (2.) That by friction they are kept there with a certain degree of force; (3.) That by the rotatory motion they tend to penetrate still further into the plane; of this we shall be convinced by observing how readily instruments designed to pierce by turning, are made to penetrate, when once introduced by means of an opening however slight. Now all this is strictly applicable to the top; the parts of the point are attached by friction; and in this way the rotatory motion is aided in effecting an entrance into the plane. This motion, moreover, being the more rapid and the better fitted to bore and press upon the surface XZ, according as the parts of the body diverge more from the axis ND in receding from the point N, it must produce the effect which takes place in the instruments abovementioned, that is, it must urge forward so much the more forcibly the parts of the point in question. From all this, it is evident that the more the figure of the top diverges from the point N, and the more rapid the rotatory motion, the greater will be the force GL compared with gravity, and consequently the more will the resultant GE tend to raise the centre of gravity above the plane. Now it is clear that in proportion as the force by which the point is supported, and at the same time the tendency of the centre to rise, become more considerable, by so much will the tendency of the axis ND toward a perpendicular to the plane be increased; so that when ND becomes vertical, it begins after a time, to incline more and more, and if the inequalities of the surface are not too great, the top will be seen to rise above the plane in very small, sudden, vertical leaps; and this we in fact observe when the point terminates in a small plane surface cut very square and perpendicular to the axis.

We have sensible proof of the truth of this explanation derived from the fact, that the pressure of the point upon the plane, is much greater than that arising from gravity simply. Indeed, when the top is put in motion upon a plane of yielding matter, the point works its way into the substance of the plane; and if it be taken into the hand, the pressure will become much more sensible than that which takes place when there is no motion.

We learn at the same time from this experiment, that the phenomenon in question requires (1.) That the point should be small compared with the distance of the parts of the top from the axis ND; and (2.) That these parts should turn with considerable rapidity; and the success will be more or less complete, as these conditions are more or less perfectly fulfilled.

It will be seen, moreover, that upon an inclined plane the top must have a tendency not to a vertical but to a perpendicular to the plane. But as it must at the same time slide along the plane, and as this motion would cause a great vacillation in passing over the inequalities of the plane, it will not so easily preserve its perpendicular position as if the plane were horizontal.

Of the Stiffness of Cords.

253. The stiffness of ropes and cords, or the difficulty with **Fig.140.** which they are bent into a given curve, is also one of the causes which diminish the effect of forces applied to machines.

In order to understand in what manner this stiffness impairs the effect of forces, let us suppose the wheel or pulley ABC to be movable about the axle I, without friction. The two weights p and q being equal, if we make a very small addition to one of them, as q, for example, no motion will follow, unless the cord $p\ ABC\ q$ be perfectly flexible. Indeed, if we imagine that this cord, instead of being perfectly flexible, is perfectly inflexible, so that the parts $A\,p$, $C\,q$, are stiff rods firmly fixed to the body of the pulley; it is evident that the pulley being moved by an external force in the direction ABC, the two weights p and q will take the situations p' and q'; but they will tend to return to their first position, and can be prevented only by the constant exertion of a particular force. If, then, the cord is neither perfectly inflexible, nor perfectly flexible, the effect of this imperfect flexibility will be, that the point A passing to A', and the point C to **Fig.141.** C', the parts $A'\ p'$, $C'\ q'$, will be a little bent, and in such a manner that the weight p' will be farther from I, and the weight q' nearer to it, than they would be if the cord were perfectly flexi-

ble ; so that a certain force is required in order to bring the parts $A'O$, CC', into the direction of tangents to the points A and C; in other words, a force must be employed which would be unnecessary but for this want of flexibility.

The pulley being always supposed to move with perfect ease upon its axle I, if instead of a cord a ribbon be employed, a very small increase in the weight q will cause the pulley to turn. But if the cord be replaced, it will evidently be necessary to augment the weight q; (1.) According as the sum of the weights p and q, or, in general, the whole force by which the cord is stretched, is more considerable; because, other things being the same, the resistance occasioned by the weights p and q, when by the stiffness of the cord they take the positions $A'O\,p'$, $CC'\,q'$, will increase as the weights themselves increase.

(2.) The addition to be made to q, must be greater according as the radius of the pulley (or of the surface over which the cord passes) is less. For the resistance which the power meets with arises from this, that the cord, instead of adapting itself to the revolving surface, remains at a certain distance, forming a curve $p'\,OA'$ and making with the surface an angle $OA'A$; and this resistance will evidently be the greater, according as the curvature $A'O$ of the cord departs more from the curvature of the surface; that is, according to the smallness of the radius of this surface.

(3.) The power applied must also be increased in proportion to the diameter of the cord. Indeed it is manifest, that, other things being the same, the cord will bend the less according as the thickness is greater; but we have just seen that the resistance to the power is greater according as the curve $A'O$ differs more from the curve $A'A$; it is therefore the greater according as $A'O$ differs less from a straight line, or the position of an inflexible rod, that is, according as the cord has a greater diameter or radius.

254. Let us suppose that k is the addition to be made to a power to render it sufficient to overcome the resistance arising from the stiffness of the cords, when the entire force by which the cord is stretched is p, the diameter of the cord which bears

the weight being δ, and the radius of the surface R. We wish to know what this addition must be, when the weight is p', the diameter of the cord δ', and the radius of the surface R'. It will be observed, after what has been said, that if there were no difference except with respect to the entire weight by which the cord is stretched, we should arrive at a solution by the proportion

$$p : p' :: k : \frac{k\,p'}{p} = \text{the addition required.}$$

But if, beside the difference in the weights, there is also a difference in the curvature of the surfaces; then, by the second of the above remarks; namely, that the additions arising from this cause are in the inverse ratio of the radii of the surfaces, we should obtain the addition in question, together with that due to a change of weight, by the following proportion,

$$R' : R :: \frac{k\,p'}{p} : \frac{k\,R\,p'}{R'\,p}.$$

Regard being had to the third remark, we shall obtain the addition to be made on account of the three causes united, by the proportion

$$\delta : \delta' :: \frac{k\,R\,p'}{R'\,p} : x = \frac{\delta'\,k\,R\,p'}{\delta\,R'\,p}.$$

The resistance in the first case, therefore, will be to the resistance in the second

$$:: k : \frac{\delta'\,k\,R\,p'}{\delta\,R'\,p}, \text{ or } :: \delta\,R'\,p : \delta'\,R\,p', \text{ or } :: \frac{\delta\,p}{R} : \frac{\delta'\,p'}{R'};$$

that is, the resistances arising from the stiffness of the cords are as the weights which stretch the cords, multiplied by the diameters of these cords, and divided by the radii of the surfaces over which they pass.

These conclusions, it may be observed, are not perfectly rigorous; but they may be regarded as sufficiently exact for practice, till experiment has thrown new light upon the subject. Indeed, experiment shows that the resistance arising from the stiffness

of cords, agrees nearly with this law; but all the experiments that have been made upon this subject have not hitherto agreed so perfectly with the theory as might be wished.

255. We shall now illustrate the foregoing principles by an example. For this purpose, let us suppose the common radius
Fig. 92. of the pulley to be 2 inches, that of the axle $\frac{1}{5}$ of the same quantity, and the diameter of the cords to be $\frac{1}{3}$ of an inch. Let us take, moreover, the result of experiments on this subject, namely, that a cord of half an inch diameter, loaded with $120^{lb.}$, and passing over a pulley of $\frac{3}{2}$ of an inch, occasions, by its stiffness, a resistance of $8^{lb.}$.

This being established, the weight p being $400^{lb.}$, as in article 242, the branches 1 and 2 will be loaded, both on account of this weight and the force added to q to overcome the friction, by a force equal to $209^{lb.}$, 6. Multiplying, therefore, this weight by $\frac{1}{3}$ of an inch, the diameter of the cord, and dividing by 2 inches, the radius common to all the pulleys; multiplying also the weight $120^{lb.}$, used in our experiment, by $\frac{1}{2}$, the diameter of the cord, and dividing by $\frac{3}{2}$, the radius of the pulley, in the experiment in question; we shall have for the results, 34, 93, and 40. Now in order to obtain the force required by the stiffness of the cord 1, 2, which passes beneath the movable pulley, we must use this proportion,

$$40 : 34, 93 :: 8^{lb.} : 6, 99 \text{ or } 7^{lb.} \text{ nearly};$$

this fourth term is the quantity which it would be necessary to add to the tension of the cord 2, if the lower pulley were fixed; but since it is movable, which diminishes the tension by $\frac{1}{2}$, as we shall see below, we have only $3^{lb.}$, 5 to be added to $109^{lb.}$, 6, by which this cord is already stretched; the whole tension will thus be $113^{lb.}$, 1.

242. We have seen that, on account of friction, the tension of the cord 3 ought to be equal to $120^{lb.}$, 7; therefore the whole force by which the cord 2, 3, which passes over the fixed block, is extended, is 113, 1 + 120, 7 on $233^{lb.}$, 8. Multiplying this force as above, by $\frac{1}{3}$, and dividing by 2, we shall have 38, 97. Seeking as before the value of the stiffness of the cord 2, 3, which passes over the fixed block, we shall have the proportion,

$$40 : 38,97 :: 8^{\text{lb.}} : 7^{\text{lb.}},79.$$

Adding now these $7^{\text{lb.}}, 79$ to $120^{\text{lb.}}, 7$, the quantity before found for the tension of the cord 3, we shall have $128^{\text{lb.}}, 5$ for the force of tension, both on account of friction and the stiffness of the cords.

The cord 4, on account of friction is stretched by a force $= 130^{\text{lb.}}, 3$; the whole force of tension upon the cord 3, 4, which embraces the movable block, is therefore $258^{\text{lb.}}, 8$; with this quantity, the dimensions of the pulleys and cords remaining the same, we shall find, as above, that the allowance for the stiffness of the cords, would be $8^{\text{lb.}}, 62$, if this cord did not embrace the movable block; but proceeding as we have done above, and for the same reason, we must take but half of this quantity; thus the tension of the cord 4, all things considered, will be $134^{\text{lb.}}, 6$ nearly. 242.

The cord 5, on account of friction, is stretched with a force $= 143^{\text{lb.}}, 5$; the whole force by which the cord 4, 5, embracing the fixed pulley, is stretched, is therefore $278^{\text{lb.}}, 1$. With this value, the dimensions remaining the same, we shall find the force required for the stiffness of the cords to be $9^{\text{lb.}}, 3$; the whole tension of the cord, therefore, is $152^{\text{lb.}}, 8$. Thus friction and the stiffness of the cords together require the weight q, which would otherwise be but $100^{\text{lb.}}$, to be $153^{\text{lb.}}$ nearly, a quantity greater by more than one half.

We have taken only half of what the calculation furnished as the quantity to be added to cords 2 and 4, on account of the stiffness. The reason of this is, that the cord by which the movable pulley is made to revolve, may be considered as turning it on a centre placed at the point where this pulley touches the other cord; and consequently the case is similar to that of a fixed pulley, having a radius equal to the diameter of the movable pulley; and since the forces required on account of the stiffness of the cords, are in the inverse ratio of the radii of the pulleys, the force in this case must be diminished one half.

Of the Balance, Steelyard, &c.

Fig. 142. 256. A *balance* is a lever of the first kind in which the fulcrum or *axis* is in the middle between the points of application of the forces or *weights*. It is apparent that with such an instrument, a state of equilibrium must indicate an equality in the weights to be compared.

There are several particulars to be attended to in the construction of a good balance. (1.) The axis should be above and not too far above the centre of gravity of the two *arms* or *beam*; for if it pass through this centre of gravity, the beam, when loaded with equal weights, or not loaded at all, will remain at rest in any position whatever; and it is intended that a horizontal position shall indicate equal weights. If the axis be below the centre of gravity of the beam, the slightest deviation of this centre from a vertical position over the axis would be followed with a motion that would tend to reverse the position of the beam. Moreover, if the axis be at a considerable distance above the centre of gravity, the beam would have too great a tendency to a horizontal position, independently of an equality in the weights, and would accordingly want the requisite sensibility.

(2.) The axis and points of application of the weights, or *points of suspension*, should be in the same straight line; otherwise the beam when loaded is liable to the defects just mentioned by having its centre of gravity shifted above the axis or too far below it. It is accordingly of great importance that the arms should be so constructed as not to bend or yield in any degree in consequence of the weights attached to them, while at the same time they should be as light as possible. To secure both these objects, in the best balances instrument-makers have given to the arms the form of hollow cones. Care is taken also to preserve the equality in point of length of the two arms, and to diminish friction, by making the axis and points of suspension of hardened steel, and in the shape of a very acute wedge, or knife-

edge, the plane surfaces with which they come in contact being likewise of the hardest substances.

257. Nevertheless, where extreme accuracy is required, the best method is to place the thing to be weighed in one of the scales, and to balance it by any convenient substances in the other scale, till the beam takes an exact horizontal position, as shown by the index *FI.** Then carefully remove the thing whose weight is sought, and replace it by accurate weights, as grains, &c., till the beam takes precisely the same position. The number of grains, &c., used will be the weight required. In this way, all error arising from the want of perfect equality in the two arms of the balance is completely obviated.

258. With all these precautions, the process of weighing is but an approximation, strictly speaking, to perfect accuracy. (1.) The best balance tends of itself with some degree of force to a horizontal position, since the slightest inclination must cause the centre of gravity to rise somewhat, and consequently, if the difference in the two weights does not exceed this force, the balance will not enable us to detect the inequality. (2.) The friction, both of the axis and of the points of suspension, is an obstacle to motion, and the difference in the weights must be sufficient to overcome this, in addition to the force just mentioned, in order that the balance may show them to be unequal.

Knowing, however, what weight is sufficient to produce motion in any case, we know what degree of accuracy we have been able to attain. This is different for different weights. A small weight can be determined more exactly, other things being the same, than a large one, since the friction increases with the pressure. A balance constructed by Ramsden for the Royal Society of London, when loaded with a weight of $10^{\text{lb.}}$, would turn with a ten-millionth part of the weight, or a little more than a 200th part of a grain; and one made by Fortin of Paris, when charged with a weight of $4^{\text{lb.}}$, would turn with a 70th part of a grain.

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* Instead of an index rising perpendicularly, a pin is sometimes attached to the end of the beam, and made to move over a nicely graduated arc provided with a microscope; and the whole apparatus enclosed under glass to prevent any agitation from the air.

Fig. 143. 259. The steelyard also is a lever of the first kind; but the fulcrum or axis of this instrument divides the distance between the points of application of the weights into unequal parts; and instead of different weights placed at the same distance, a constant weight or *poize* is placed at different distances to effect an equilibrium with the article to be weighed; and the weight of this article will be to that of the poize inversely as their distances from the axis. When these distances are equal, the weight of the poize is equal to that of the article weighed. At double this distance, the poize indicates a weight double its own, and at half this distance the weight indicated is half that of the poize, and so on. The longer arm of the steelyard, therefore, being graduated upon this principle, it becomes a convenient and sufficiently accurate instrument for weighing all gross substances.

The same rule is to be observed with respect to the position of the axis of motion in the construction of the steelyard, as in that of the balance; that is, the axis must be in a line with the points of suspension, and a little above the centre of gravity, so that the arms when unloaded, or when equal moments are indicated, shall preserve a horizontal position.

Fig. 144. 260. In the *bent-lever balance* represented in figure 144, instead of a movable weight resting upon a horizontal beam, the natural weight of an inclined arm is made use of, and is drawn out to different distances from a vertical position according to the weight attached to the other arm. By applying different known weights to the arm A, and graduating the arc CD according to the positions of the arm B, we shall have an instrument very analogous to the steelyard; for the weight B may be considered as shifted to different distances along the horizontal line FD, since it would have precisely the same effect here as in its actual position. Consequently, if the direction of the weight attached to the arm A preserved always the same horizontal distance from the axis of motion (which it might be made to do, by suspending it from a single cord applied to the arc of a circle having the axis for its centre,) the arc CD might be graduated by dividing the radius DF into equal parts in the manner of the steelyard, and then transferring these divisions with the numbers denoting the weights to the arc CD, by means of vertical lines.

261. Sometimes the axis is made to carry a wheel, the teeth of which act upon a pinion furnished with an index and dial-plate, whereby a slight motion is rendered very conspicuous.

The bent-lever balance is better adapted to despatch than any other instrument for weighing. But it is liable to irregularity, and is not susceptible of the accuracy of the balance.

DYNAMICS.

Of Motion uniformly accelerated.

262. A body, perfectly free, having once received an impulse will continue its motion, the velocity and the direction remaining 17.
always the same as at the first instant. But if it receive a new impulse in the same direction, or in a direction opposite to the first, it will move with a velocity equal in the former case to the sum, and in the latter to the difference, of the velocities it has successively received.

Hence, if we suppose that, at determinate intervals of time, the body receives new impulses in the same direction, or in a direction opposite to the first, it will have a varied or unequal motion, and its velocity will change or become different at the commencement of each interval of time.

Whatever this may be, the velocity of a body at the end of any given period of time, is to be estimated by the space which this body is capable of describing in a unit of time on the supposition that its motion becomes uniform at the instant from which this velocity is to be reckoned.

Any force, which acting upon a body causes it to vary its motion, is called an *accelerating* or *retarding* force. When it acts equally at equal intervals of time, it is called a *uniformly accelerating*, or *uniformly retarding force*, according as it tends to increase or diminish the actual velocity of the body.

We shall now examine the circumstances of motion uniformly accelerated.

263. Since in this kind of motion, the accelerating force acts always in the same manner, if we suppose that u is the velocity

communicated at each unit of time, it is evident that the successive velocities of the body will be u, $2u$, $3u$, &c., so that after a number of units of time, denoted by t, the velocity acquired will be u, taken as many times as there are units in t; that is, it will be $t \times u$ or $t u$.

264. Hence, (1.) In the case of motion uniformly accelerated, the number of degrees of velocity which the body acquires, increases as the number of intervals or periods during which the motion continues, which may be expressed by saying that for different times *the velocities acquired are as the times elapsed from the commencement of the motion.*

Thus, if we call v the velocity that the body acquires at the end of the time t, we shall have

$$v = u t.$$

(2.) The velocities which the body will be found to have during the lapse of each successive interval, will form an arithmetical progression or progression by differences,

$$\div u \,.\, 2u \,.\, 3u \,.\, \&c.$$

the last term of which is $u t$ or v, and of which the number of terms is t, that is, is denoted by the number of actions of the accelerating force.

(3.) Also, since the velocities u, $2u$, &c., are simply the spa-
22. ces that the body would describe in the corresponding intervals of time respectively; the whole space described during the time t will be the sum of the terms $u + 2u +$ &c., of this progression,
Alg.229. that is, it will be expressed by $(u + v) \times \frac{1}{2} t$. Therefore, if we call s this whole space passed over from the commencement of the motion, we shall have

$$s = (u + v) \times \tfrac{1}{2} t.$$

265. Let us suppose now, that the accelerating force acts without interruption, or, which amounts to the same thing, that the time is divided into an infinite number of infinitely small parts, which we shall call instants; and that at the beginning of each instant, the accelerating force exerts a new impulse upon the body. Let us suppose, also, that it acts by degrees infinitely small.

Then, u being infinitely small with respect to v, which is the velocity acquired during the infinite number of instants denoted by t, u is to be neglected in the equation $s = (u + v) \times \frac{1}{2} t$ which gives Cal. 5.

$$s = \frac{1}{2} v t.$$

266. This being established, let us suppose that at the end of the time t, the accelerating force ceases to act; the body will 17.
continue its motion with the velocity v that it had acquired; that is, in each unit of time, it will describe a space equal to v; according- 22.
ly, if it were to continue with this velocity during the time t, it would describe a space equal to $v \times t$, that is, double the space s or $\frac{1}{2} v t$, described in the same time by the successive action 265.
of the accelerating force. Therefore, *in motion uniformly and continually accelerated, the space described during a certain time is half of that which the body would describe in an equal time with the last acquired velocity continued uniformly.*

267. Since the acquired velocity increases with the times elapsed, if we call g the velocity acquired at the end of a second, the velocity acquired after a number t of seconds, will be $g t$; that 264.
is, we have,

$$v = g t;$$

and accordingly the equation $s = \frac{1}{2} v t$, found above, becomes

$$s = \frac{1}{2} g t \times t = \frac{1}{2} g t^2.$$

If, therefore, we represent by s' another space described in the same manner during a time t', we shall have, according to the above reasoning,

$$s' = \frac{1}{2} g t'^2,$$

from which we deduce the proportion,

$$s : s' :: \frac{1}{2} g t^2 : \frac{1}{2} g t'^2 :: t^2 : t'^2;$$

we hence learn that, with respect to motion uniformly accelerated, *the spaces described are as the squares of the times.*

268. Moreover, since the velocities are as the times, we con- 264.
clude also, *that the spaces described are as the squares of the velocities.*

269. Therefore *the velocities and the times are each as the square roots of the spaces described from the commencement of the motion.*

270. The principles here established are equally applicable to the case of motion uniformly retarded, provided that by the times we understand those which are to elapse, and by the spaces those which are to be described, from the instant in question till the velocity is destroyed.

271. From the equation $s = \frac{1}{2} g t^2$, the quantity g, by which we have understood the velocity that the accelerating force is capable of producing by its action, exerted successively during a second of time, is what we call the accelerating force, since we must judge of this force by the effect which it is capable of producing in a body in a determinate time, an effect which is nothing else but the communi-
27. cation of a certain velocity.

Of free Motion in heavy Bodies.

272. It is the kind of motion we have been considering, to which the motion of heavy bodies is to be referred. But before applying to this subject the theory above developed, it will be proper to make known a few facts concerning gravity, in addition
74. to those heretofore given.

As to the magnitude of the force of gravity, it is different, strictly speaking, in different latitudes and at different distances from the centre of the earth in the same latitude. But the quantities by which it varies, as we depart from the equator, are very small, and do not in any manner concern us at present. The same may be said of the variations it undergoes, according as we rise above, or descend below, the mean surface of the earth; they cannot become sensible, except by changes of distance much more considerable than any to which we are accustomed; so that for the present we may regard gravity as a force every where the same, or one which urges bodies downward by the same quantity in the same time.

This force is to be considered also as acting, and acting equally at each instant, upon every particle of the matter about

us. Now it is evident, that if each of the parts of a body receive the same velocity, the whole will move only with the velocity that any detached portion would have received; so that the velocity which gravity impresses upon any mass whatever, does not depend upon the magnitude of this mass; it is the same for a large body as for a small one. It is true, however, that all bodies are not observed to fall from the same height in the same time; but this difference is the effect of the resistance of the air, as we shall see hereafter; and when bodies are made to fall in close vessels, from which the air has been withdrawn, though of very different masses, they are found to descend through the same space in the same time.

It may be well to notice here the distinction between the effect of gravity and that of weight. The effect of gravity is to cause, or tend to cause in each part of matter, a certain velocity, which is absolutely independent of the number of material particles. But weight is equal to the effort necessary to be exerted, in order to prevent a given mass from obeying its gravity. Now this effort depends upon two things; namely, the velocity that gravity tends to cause in each part, and the number of parts on which this force is exerted. But as the velocity which gravity tends to give, is the same for each part of matter, the effort to be exerted in order to prevent a given mass from obeying its gravity, is proportional to the number of parts, that is, to its mass. *Thus weight depends upon the mass, whereas gravity has no relation to it.*

273. Having made known these particulars with regard to gravity, we proceed to the laws of motion of heavy or gravitating bodies.

Since gravity acts equally and without interruption, at whatever distance the body is from the surface of the earth (at least, so far as our experience extends), gravity is a uniformly accelerating force, which at each instant causes in a body a new degree of velocity, that is always the same for each equal instant; so that the velocities acquired, increase as the times elapsed; the spaces passed over are as the squares of the times, or as the squares of the velocities; the velocities are as the square roots of the spaces described; the times are also as the square roots of the spaces described; in short, all that we have said respect-

ing a uniformly accelerating force, is strictly applicable to gravity, it being well understood at the same time, that the resistance of the air and obstructions of every kind are out of the question.

In order to determine, therefore, with respect to the motion of heavy bodies, the spaces described and the velocities acquired, we require only one single effect of gravity for a determinate time. For the equations $v = g\,t$, $s = \frac{1}{2} g\,t^2$, enable us to calculate each of the particulars above enumerated, when the value of g is known.

It must be recollected that by g we have understood the velocity which a body acquires by gravity in one second of time.
267. Now we know by actual observation, that a body, not impeded by the resistance of the air or other obstacle, falls at the surface of the earth through 16,1 feet in one second. We shall see hereafter how this is determined.

But we have shown that with the velocity acquired by a series of accelerations, the body would describe with a uniform
266. motion double the space in the same time. Hence the velocity acquired by a heavy body at the end of the first second of its fall is such, that if gravity ceased to act, it would describe twice 16,1 feet, or 32,2 in each succeeding second. Therefore $g = 32{,}2$ feet.

274. Now of the two equations $v = g\,t$, and $s = \frac{1}{2} g\,t^2$, the first teaches us that in order to find the velocity acquired by a heavy body, falling freely, during a number t of seconds, it is necessary to multiply the velocity acquired at the end of the first second by the time t, or number of seconds.

Hence, *when a heavy body has fallen during a certain number of seconds, the velocity acquired is such, that if gravity ceased to act, the body would describe in each second as many times* 32,2 *feet, as there were seconds elapsed.*

Thus a body that has fallen during 7 seconds, will move at the end of the 7 seconds, with a velocity equal to 7 times 32,2 or 225,4 feet in a second without any new acceleration.

275. From the second of the above equations, namely,

$$s = \tfrac{1}{2} g\,t^2,$$

we learn that in order to find the space or height s through which a heavy body falls in a number t of seconds, we have only to multiply the square of this number of seconds by $\frac{1}{2}g$, that is, by the space described in the first second.

Hence, *the height or number of feet through which a heavy body falls during a number* t *of seconds is so many times* 16,1 *feet as there are units in the square of this number of seconds.*

Thus, when a body has been suffered to fall freely during 7 seconds, we may be assured that it has passed through a space equal to 49 times 16,1 feet, or 788,9 feet. We see, therefore, that when, in the case of falling bodies, the time elapsed is known, nothing is more easy than to determine the velocity acquired, and the space described.

276. If the question were to find the time employed by a body in falling from a known height, the equation $s = \frac{1}{2}g\,t^2$,

$$\text{gives } t^2 = \frac{s}{\frac{1}{2}g}, \text{ and consequently,}$$

$$t = \sqrt{\frac{s}{\frac{1}{2}g}};$$

that is, we seek how many times the height s contains $\frac{1}{2}g$, or 16,1 feet, the space described by a body in the first second of its fall, and take the square root of this number.

277. If we would know from what height a heavy body must fall to acquire a given velocity, that is, a velocity by which a certain number of feet is uniformly described in a second; from the equation $v = g\,t$, I deduce the value of t, namely, $t = \frac{v}{g}$; substituting this value in the equation $s = \frac{1}{2}g\,t^2$, I have

$$s = \tfrac{1}{2}g \times \frac{v^2}{g^2} = \frac{v^2}{2g},$$

by which I learn, that in order to find the height s from which a heavy body must fall to acquire a velocity v, of a certain number of feet in a second, the square of this number of feet is to be divided by double the velocity acquired by a heavy body in one second, that is, by 64,4.

Thus, if I would know, for example, from what height a heavy body must fall, to acquire a velocity of 100 feet in a second, I divide the square of 100, namely, 10000, by 64,4; and the quotient $\frac{10000}{64,4} = 155,2$ &c., is the height through which a body must fall to acquire a velocity of 100 feet in a second.

We might evidently make use of the same formula in determining to what height a body would rise, when projected vertically upward with a known velocity.

Moreover, from the above equation, $s = \frac{v^2}{2g}$ we obtain

$$v^2 = 2\,g\,s, \text{ or } v = \sqrt{2\,g\,s} = 8{,}024\,\sqrt{s},$$

that is, the velocity acquired in falling through any space s, is equal to $\sqrt{2\,g\,s}$, or equal to eight times the square root of s nearly, v, g, and s, being estimated in feet. Thus the velocity acquired in falling through 1 mile or 5280 feet, is equal to

$$8{,}024\,\sqrt{5280} = 583 \text{ feet very nearly.}$$

278. By these examples it will be seen that all the circumstances of the motion of heavy bodies may be easily determined; and it is accordingly to these motions, that we commonly refer all others; so that, instead of giving immediately the velocity of a body, we often give the height from which it must fall to acquire this velocity. Occasions will be furnished for examples hereafter.

We will merely observe, therefore, by way of recapitulation, that all the circumstances of accelerated motion, and consequently of the motion of heavy bodies, are comprehended in the two equations $v = g\,t$, $s = \frac{1}{2}\,g\,t^2$; so that, g being known, and one of the three things, t, s, v, or the time, space, and velocity, the two others may always be found, either immediately by one or the other of the above equations, or by means of both combined after the manner of article 277.

279. When a body is subjected to the action of a force that is exerted upon it without interruption, but in a different manner at each successive instant, we give to the motion the general denomination of *varied*. We have examples of varied motion in the unbending of springs; although in this case the velocity goes

on increasing, still the degrees by which it increases go on diminishing. The same may be observed with respect to the degrees by which the motion of a ship arrives at uniformity; the action of the wind upon the sails diminishes according as the ship acquires motion, because it is withdrawn so much the more from this action, according as it has more velocity.

280. The principles necessary for determining the circumstances of this kind of motion are easily deduced from the principles that we have laid down with regard to uniform motion, and motion uniformly accelerated.

In whatever manner motion is varied, if we consider it with respect to instants infinitely small, we may suppose that the velocity does not change during the lapse of one of these instants. Now, when the motion is uniform, the velocity has for its expression the space described during any time t, divided by this time. Accordingly, when the motion is uniform only for an instant, the velocity must have for its expression the infinitely small space described during this instant divided by this instant. Hence, if s represents the space described, in the case of a variable motion, during any time t, $d\,s$ will represent the space uniformly described during the instant $d\,t$; we have, therefore,

$$v = \frac{d\,s}{d\,t} \text{ or } d\,s = v\,d\,t,$$

as the *first fundamental equation of varied motion.*

281. In the equation $v = g\,t$, we have understood by g the velocity which the accelerating force is capable of giving to a body in a determinate time, as one second, by an action that is supposed to continue constantly the same. In the equation

$$d\,v = g\,d\,t,$$

the same thing is to be understood. But we must observe that the accelerating force being supposed to be variable, the quantity g which represents the velocity that the accelerating force is capable of producing, if it were constant for one second, this quantity g, I say, is different for all the different instants of the motion. Indeed, it will be readily conceived, that when the accelerating force becomes less, the velocity that it is capable

of generating in a second by its action repeated equally during eac instant of this second, must be less, and *vice versâ.*

282. From the two equations $d\,s = v\,d\,t$, $d\,v = g\,d\,t$, we can obtain a third that may be employed with advantage. Thus from the equation $d\,s = v\,d\,t$, we deduce $d\,t = \frac{d\,s}{v}$; substituting this value instead of $d\,t$ in the equation $d\,v = g\,d\,t$, we have,

$$d\,v = g \times \frac{d\,s}{v},$$

or,

$$g\,d\,s = v\,d\,v.$$

283. We remark, that in the process by which we have just arrived at the equation $d\,v = g\,d\,t$, we regarded the velocity as increasing. If it had gone on diminishing, it would have been necessary, instead of $d\,v$ to put $-d\,v$; so that the two equations $d\,v = g\,d\,t$, and $g\,d\,s = v\,d\,v$, to become general, must be written

$$\pm\,d\,v = g\,d\,t, \qquad \pm\,g\,d\,s = v\,d\,v,$$

the upper sign being used when the motion is accelerated, and the lower when the motion is retarded.

284. There is a fourth equation that may be deduced from the two fundamental equations, and which should not be omitted. Thus, the equation $d\,s = v\,d\,t$ gives $v = \frac{d\,s}{d\,t}$; whence we obtain

$$d\,v = d\left(\frac{d\,s}{d\,t}\right);$$

substituting this value for $d\,v$ in the equation $g\,d\,t = \pm\,d\,v$, we have

$$g\,d\,t = \pm\,d\left(\frac{d\,s}{d\,t}\right).$$

If we suppose, as we are authorized to do, that $d\,t$ is constant, we shall have,

$$g\,d\,t = \pm\,\frac{d\,d\,s}{d\,t} \text{ or } g\,d\,t^2 = \pm\,d\,d\,s.$$

But it must be recollected that, in the equation $g\,d\,t^2 = \pm\,d\,d\,s$, it is supposed that $d\,t$ is constant. When $d\,t$ is variable, we make use of the equation

$$g\,d\,t = \pm\,d\left(\frac{d\,s}{d\,t}\right).$$

Occasions will occur in which these formulas will be of great use. But we must not forget that the quantity g which they contain, represents, for each instant, the velocity which the accelerating force is capable of giving to the moving body in a known interval of time, as one second, if during the second it were to act with a uniformly accelerating force; so that as each quantity g measures, for each instant, the effect of which the accelerating force is capable, we shall give it, for brevity's sake, the name of accelerating force.

Of the direct Collision of Bodies.

285. We suppose, in what follows, that no account is taken of the gravity of bodies, of friction, or other resistance.

We suppose also that the bodies, whose collision is the subject of consideration, act the one upon the other according to the same straight line, passing through their centres of gravity, and that this straight line, is perpendicular to the plane touching their surfaces at the point where they meet.

We shall consider bodies as divided into two classes, denominated *unelastic* and *elastic*; the former are supposed to be such that no force can change their figure; the latter are regarded as capable of having their figure changed, that is, of being *compressed*, but as endued at the same time with a property by which this figure is resumed after the compressing force is removed.

Although there are not in nature bodies of a sensible mass, that answer perfectly to each of these descriptions, yet it is only by proceeding upon such suppositions, that we are able to determine the action of such bodies as are actually presented to our observation.

Of the direct Collision of unelastic Bodies.

286. Two unelastic bodies which meet, (or one of which falls upon the other at rest,) communicate, or lose, a part of their motion; and in whatever manner this takes place, we may always, at the instant of the collision, represent each body, according to the principle of D'Alembert, as urged with two velocities, one of which remains after the collision, while the other is de-
133. stroyed.

287. Let us, in the first place, suppose the two bodies to move in the same direction. That which goes the faster, will evidently lose a part of its velocity by the collision, and the other will gain by it. Let m be the mass of the impinging body, and u its velocity before collision; n the mass of the impinged body (which may be less or greater than m), and v its velocity before collision. Let us suppose that the velocity u changes to u' by the collision; m will accordingly have lost $u - u'$. I will consider m as having, at the instant of collision, the velocity u', and the velocity $u - u'$. If we suppose, in like manner, that v becomes v' by the collision, n will have gained $v' - v$; I can accordingly consider it, at the instant of collision, as having the velocity v', in the direction of the actual motion, and the velocity $v' - v$ in the opposite direction; since, on this supposition, it will really have only the velocity $v' - (v' - v)$ or v.

As, therefore, among these four velocities there can, by supposition, remain only u' and v', the two others $u - u'$, and $v' - v$, must be destroyed in the act of collision. Now as these are directly opposite, it is necessary that the quantities of motion, which the bodies would have in virtue of these velocities, should be equal; we have, therefore,

33.
$$m(u - u') = n(v' - v).$$

Now in order that u' and v' may, as we have supposed, be the velocities which the two bodies m and n have after collision, these velocities must be such, that the impinging body shall not have the greater action over the impinged, that is, that the two bodies shall, after collision, proceed in company; we have, accordingly, $v' = u'$; and hence form the equation,

$$m(u - u') = n(u' - v),$$

or,

$$mu - mu' = nu' - nv,$$

we obtain

$$u' = \frac{mu + nv}{m + n};$$

therefore, *when the bodies move in the same direction, in order to find the velocity after collision, we take the sum of the quantities of motion, which the bodies had before collision, and divide this sum by the sum of the masses.*

Thus, if m, for example, be equal to 5 ounces and n to 7, u equal to 8 feet in a second, and v to 4 feet in a second, we shall have,

$$u' = \frac{5 \times 8 + 7 \times 4}{5 + 7} = \frac{40 + 28}{12} = \tfrac{68}{12} = 5\tfrac{2}{3};$$

that is, the velocity after collision will be five feet and two thirds in a second.

288. If one of the two bodies, as n for example, were at rest before collision, we should have $v = 0$, and the expression of the velocity after collision would accordingly become

$$u' = \frac{mu}{m + n};$$

that is, we should divide the quantity of motion belonging to the impinging body by the sum of the masses of the two bodies.

If, however, instead of deducing this case from the more general one, we would find it directly, we should proceed according to the same principles, and consider the impinged body as having, in consequence of the collision, a velocity u', equal to and in the direction of that which it is to have after collision, and a velocity $-u'$, of the same magnitude, but in the opposite direction. Thus, since it is to preserve only the first, it is necessary that in virtue of the second it should be in equilibrium with the body m, having a velocity $u - u'$ which it is to lose. Accordingly we must 133.
have

$$m(u - u') = n u',$$

from which we deduce

$$u' = \frac{mu}{m+n},$$

the same as the expression above obtained from the general formula.

289. When the bodies move in opposite directions, in order to find the velocity after collision, it is only necessary to suppose in the first formula, that v is negative, which gives

$$u' = \frac{mu - nv}{m+n};$$

that is, *when the bodies move in opposite directions, in order to find the velocity after collision, we take the difference of the quantities of motion belonging to the bodies before collision, and divide by the sum of the masses; and this velocity will take place in the direction of that body which had the greater quantity of motion.*

We might also obtain this result directly by proceeding as in the above example.

Thus the laws of the direct collision of unelastic bodies reduce themselves in all cases to this single rule; *the velocity after collision is equal to the sum or to the difference of the quantities of motion before collision (according as the bodies move in the same or in opposite directions), divided by the sum of the masses.*

Of the Force of Inertia.

290. We have supposed in what we have said, that independently of gravity, the resistance of the air, and other obstacles, one of the two bodies opposes a resistance to the other, and makes it lose a part of its velocity. But how can a body without gravity, and which is confined by no obstacle, oppose a resistance? Does not this seem to imply that it would be capable of giving motion?

Now every resistance does not always imply an actual motion in the resisting body. If the body A, for example, be drawn at the same time by two equal and opposite forces represented by AB, AC, it would evidently have no motion. But it is not less evident Fig.145. that if a force equal to CA were to act upon it in the direction CB, this force would be destroyed by the effort AC, and the body would yield in virtue of the force AB, equal to that just applied.

We do not pretend to decide whether the resistance which bodies oppose to motion, does or does not arise from a cause of this kind. However the fact may be, the resistance in question which we call the *force of inertia*, differs from the resistance opposed by active forces (as that of bodies which impinge against each other in opposite directions) in this, that these last annihilate a part of the motion; whereas, with respect to the force of inertia, while it destroys a part of the motion in the impinging body, this motion passes wholly into the impinged body, as is clearly shown by the equation

$$m\,(u - u') = n\,(u' - v),$$

above obtained for determining the motion after collision of two bodies which move in the same direction; for $u - u'$ is the ve- 287.
locity lost by the impinging body, and consequently $m\,(u - u')$ is the quantity of motion which this body loses by collision. We have, in like manner, seen, that $u' - v$, is the velocity, and $n\,(u' - v)$ the quantity of motion, gained by the impinged body. Now we have shown that these two quantities must necessarily be equal.

The force of inertia, therefore, is, properly speaking, the means of the communication of motion from one body to another. Every body resists motion, and it is by resisting that it receives motion; it receives also just so much as it destroys in the body that acts upon it.

We hence see that, every obstacle being removed, however small we suppose the impinging body, and however great the mass impinged, motion will always take place upon collision. When, for example, one of the two bodies is at rest, the velocity which has for its expression

$$u' = \frac{m\,u}{m + n}$$ 288.

can never become zero, whatever be the values assigned to m, n, and u; the only case where u' can be infinitely small, is that in which n is infinitely great. Thus, if in nature we see bodies lose the motion that they have received, it is because they communicate to the material parts of the bodies which surround them. Now it is evident from the formula,

$$u' = \frac{m\,u}{m+n},$$

that the greater the mass of the impinged body n, the less (other things being the same) will be the velocity u', n being considered as the sum of the material particles among which m parts with its motion. It will be seen, therefore, that the velocity u' may soon be so reduced as to escape the notice of the senses, even when it is not opposed by immovable obstacles, as friction, &c.

291. The force of inertia, being a force peculiar to matter, exists equally in every equal portion of matter, and consequently in a determinate mass it takes place according to the quantity of matter, or in proportion to the mass; and as the mass is proportional to the weight, the force of inertia may be regarded as proportional to the weight. But we must take care not to infer hence, that the force of inertia arises from gravity; it is altogether independent of it; indeed, if while a body is falling freely, it be forced forward by the hand with a velocity greater than that of its natural descent, the hand will experience, on overtaking the body, a blow or resistance, that manifestly cannot be attributed to gravity, which acts only downward. Still less can it be ascribed to the resistance of the air; for the resistance of the air, being capable of acting only on the surfaces of bodies, cannot, like the force of inertia, be proportional to the quantity of matter.

The force of inertia, therefore, is a force peculiar to matter, by which every body resists a change of state, as to motion and rest. *The force of inertia is proportional to the quantity of matter, and takes place in all directions according to which an effort is made to move a body.*

Application of the Principles of Collision of unelastic Bodies.

292. The principles which we have laid down respecting the collision of unelastic bodies are applicable, whether the bodies impinge directly upon each other, as we have supposed, or whether they act upon each other by means of a rod which joins their centres of gravity, or whether one draws the other by a thread, provided the action is immediately and perfectly transmitted to the centre of gravity of each.

If, for example, the two bodies m and n act upon each other Fig. 146.
by means of a thread passing over a pulley P, and we would determine the motion that they would receive in virtue of their gravity, we observe that gravity tends to impress the same velocity upon each of the two bodies at each instant. Now as one cannot 74.
move without drawing the other, the same thing will take place with regard to the two bodies at each new action of gravity, as if the two bodies drew each other in opposite directions with equal velocities; therefore, in order to find the resulting velocity, it is necessary, calling u the velocity produced by gravity at each instant in a free body, to take the difference $m\,u - n\,u$ of the quantities of motion, and to divide it by the sum $m + n$ of the masses; we have accordingly

$$\frac{m\,u - n\,u}{m + n} \quad \text{or} \quad \frac{m - n}{m + n}\,u$$

for the actual velocity that each new action of gravity would give to the body m. We see, therefore, since m, n, and u, are constant quantities, that the body m is carried with a motion uniformly accelerated, and that the force which actually accelerates it, is to free gravity 28.

$$:: \frac{m - n}{m + n}\,u : u :: \frac{m - n}{m + n} : 1 :: m - n : m + n.$$

Consequently, if we call g the velocity which gravity communicates to a free body in one second, we shall have that which it would communicate in the same time to the body m, impeded by the body n, by the proportion

$$m + n : m - n :: g : \frac{m - n}{m + n} g.$$

If, therefore, we call w the velocity of m at the expiration of a
264. number t of seconds, we shall have

$$w = \frac{m - n}{m \times n} g\, t;$$

267. and the space which it will have described, will be

$$s = \frac{m - n}{m + n} \times \tfrac{1}{2} g\, t^2;$$

which is readily found, by putting for t the given number of sec-
276. onds, and for g 32,2 feet.

293. If at the first instant the body n, supposed to have less mass than the other, receive an impulse or velocity v, that is, if it were struck in such a manner, that, being considered free and without gravity, it would pass over in a second a number of feet denoted by v, it would divide this action with the body m which it would draw during a certain time. In order to determine how the action in question would be divided, it must be remarked, that at the first instant the action of gravity being infinitely small or nothing, the body n, urged with a velocity v, acts upon the body m as if this last were at rest. It is necessary, therefore, in order to find the velocity remaining after the action, to divide the
28. quantity of motion $n\, v$ by the sum of the masses, which gives $\frac{n\, v}{m + n}$ for the velocity with which n would draw m, if gravity did not act in the following instants. But as we have seen that it would act in such a manner as to give to the body m, in the opposite direction, the velocity $\frac{m - n}{m + n} g\ t$ in the time t; it follows that, at the expiration of the time t, the body n will have only the velocity

$$\frac{n\, v}{m + n} - \frac{m - n}{m + n} g\, t.$$

Whence it will be seen, that however small n may be, and however small the velocity v, and however considerable the mass of the

body m, n will always draw m for a certain time, after which m will prevail, and draw n in its turn.

Indeed, whatever may be the quantity of motion $n\,v$, impressed upon n, so long as it is of a finite value, it is evident that it would always be necessary, in order to counteract it, that gravity should act for a certain time, for it only acts by degrees infinitely small at each instant.

If we would know at the expiration of what time m will cease to ascend, we should proceed thus. Let t' be the time employed, by a heavy body, falling freely, in acquiring the velocity v; according to article 263, we shall have

$$v = g\,t';$$

therefore the velocity of n will be changed to

$$\frac{n\,g\,t'}{m+n} - \frac{m-n}{m+n}\,g\,t;$$

which being put equal to zero, gives

$$n\,g\,t' = (m - n)\,g\,t,$$

from which we deduce

$$t = \frac{n\,t'}{m-n}.$$

If, for example, the velocity v, supposed to be impressed upon n is such as a heavy body would acquire in one second, we should have $t' = 1''$. Suppose $m = 100^{\text{lb.}}$, and $n = 1^{\text{lb.}}$, we should have

$$t = \frac{1''}{100-1} = \frac{1''}{99},$$

that is, the body n would draw the body m only during one ninety-ninth of a second; still it would draw it.

We see, therefore, that there is not a finite force, however small, which is not capable of overcoming the weight of a body; and that it is not possible for a body actually in motion, to be placed in equilibrium with the weight of another body, that is, with a body that has the simple tendency of gravity. The former

would first draw the latter, and afterward be drawn by it; there would indeed be an instant of rest, but it would be that in which the former had lost all the velocity impressed upon it, and this state would continue only for an instant.

294. Thus the force of bodies in motion cannot be estimated by weights, that is, by the simple tendency of gravity in bodies destitute of local motion; but only by other forces of the same kind, as those of heavy bodies having fallen from a certain height. Hence, in order to have an idea of the force of a body of 3 pounds, carried with a velocity of 50 feet in a second, I should seek by the method of article 277, from what height a heavy body must fall to acquire a velocity of 50 feet in a second, and I should find it to be 38,8 feet nearly. I should conclude, therefore, that a body of 3 pounds, urged with a velocity of 50 feet in a second, must strike as if it had fallen from a height of 38,8 feet.

295. The force which bodies in motion are capable of exerting, is called *percussion*.

The force of percussion cannot, therefore, in any way be compared with simple pressure, or the effort which a mass is capable of making by its weight without local motion. A blow of a hammer, though feeble, will drive a nail into a block of wood; also a body of small mass, which by its fall had acquired but little velocity, would be attended with the same result, while a very considerable weight would produce no effect.

The reason of this difference is, that in the former case, all the degrees of velocity possessed by the body in motion, are exerted in an instant; whereas in the latter, the weight, which exerts only a pressure, receives its degrees of force successively, and imparts them in the same manner to the nail and the surrounding mass; and as each of these degrees is infinitely small, it is absorbed as soon as it is received.

Of the Collision of elastic Bodies.

296. Although elastic bodies, according to the definition which we have given, must be compressible, we are not hence to infer that they must be so much the more compressible, as they are more elastic. A ball of wool, for example, is not more elastic than a ball of ivory, although it is much more compressible.

Be this as it may, compressibility seems to be inseparable from elasticity. A body in virtue of its compressibility, changes its figure, when a force is applied to it from without; and in virtue of its elasticity, it tends to recover this figure. But among all elastic bodies, some recover their figure entirely, others only in part. These last are called *imperfectly elastic bodies.* As to the former, they may resume their figure more or less promptly, and by very different degrees. But if they are such that, after being struck, they restore themselves according to the same degrees by which they were compressed, we call them *perfectly elastic bodies.* In other cases they are denominated simply *elastic bodies.* We shall here consider only those that are perfectly elastic.

We observe with respect to perfectly elastic bodies, that in collision, a resistance takes place on the part of the body which has the least velocity, and consequently a compression, and that on this account not only a restoration of the figure follows this compression, but this restoration is itself followed by a new change of figure directly contrary to the first. To this succeeds another, which reduces the body to the figure first given by the compression, and so on. In this way the parts of each body have, with respect to their centre of gravity, a vibration, or motion backward and forward; since the parts tend to return to their first figure by a motion which goes on increasing, and thus carries them beyond their former position. These changes of figure, which alternate with each other, are sensible in several elastic bodies, when struck, and particularly in those that are of the sonorous class.

It is not, however, to be supposed, that these vibrations affect the velocity which the bodies take after collision. They can
134. have no influence upon the motion of the centre of gravity, since this motion takes place in each of the two bodies independently of the other.

The collision of perfectly elastic bodies is to be viewed, therefore, in the following manner. When the two bodies m, n,
Fig. 148. come to meet in C, the resistance which n opposes to m, causes them to be mutually compressed, until the two centres and the point of contact have all the same velocity; thus far every thing takes place, as in the collision of hard bodies, with the exception of the change of figure, which can contribute nothing to the quantity of motion lost or gained.

The change of figure is effected in such a manner, that each of the two bodies is flattened to the same degree on opposite sides; since the parts farthest removed from contact, advancing more rapidly in the one body and less rapidly in the other, until the compression is completed, crowd very much the intermediate parts. The compression once finished, the parts of each body bordering upon the points of contact, support themselves the one against the other, while the contact is transferred; and the recoil of the spring takes place toward the parts opposite to the point of contact, with all the force by which the bodies tend to restore their figure.

It will accordingly be seen that the impinging body loses by the recoil, a velocity equal to that which it had lost by the compression; and that, on the other hand, the impinged body gains by the recoil a velocity equal to that which it had gained during the compression; and, although the two bodies do not cease to exert their elastic force when they have regained their original figure, they have no longer any action upon each other, since, the force with which they go on to dilate themselves beginning now to grow less, they separate from each other at this conjuncture.

If, when the two bodies move in the same direction, u is the velocity of the impinging body, and v that of the impinged; u' being supposed the common velocity which they would have

after collision, considered as unelastic, $u - u'$ would be the velocity lost by the impinging body; when, therefore, the recoil of the spring (taking place in a direction opposite to the motion) causes as much motion to be lost, as had already been lost by the compression, there will remain only the velocity

$$u - 2(u - u') = 2u' - u.$$

As to the impinged body, $u' - v$ is the velocity gained by collision; and we have seen that, by the recoil of the spring, it acquires as much more; it will have, therefore,

$$v + 2(u' - v) = 2u' - v.$$

This case comprehends that in which one of the two bodies is at rest before collision.

If the bodies move in opposite directions, the reasoning is precisely the same for the one which has the greater quantity of motion. As to the other, it would, considered as unelastic, lose its velocity by the collision, and acquire another in the opposite direction; u' being this velocity, we shall have $v + u'$ for the velocity lost. Doubling this effect, on account of the bodies being elastic, and adding it to the original velocity $-v$, we have

$$2(v + u') - v = 2u' + v.$$

297. By attending to the resulting expression in each of the above cases, it will be seen that the circumstances of the collision of bodies perfectly elastic are all comprehended in this single rule;

Seek the common velocity which the two bodies would have after collision, if they were destitute of elasticity; then from double this velocity, take the velocity which each had before collision, and we shall have the velocity of each after collision; it being understood that when the bodies move in opposite directions before collision, the sign $-$ is to be given to the velocity of that body which has the less quantity of motion.

298. From the principles above laid down, we might easily obtain, for the collision of elastic bodies, formulas which should contain only the masses and velocities before collision. In order

to this, it would only be necessary to substitute, in the expressions $2u' - u$ and $2u' \pm v$, the value of u' furnished by the rules of articles 286, 288. But as these formulas would not present themselves in a manner so easy to be retained as the rules we have given, we leave this substitution to be made by those who may wish to see the result.

299. We observe that, when one of the two bodies is at rest, the velocity which it would receive by the collision is double that which it would have had, considered as non-elastic. This is an evident consequence of the general rule.

300. To give a few examples of these rules, let us suppose, in the first place, that the two bodies are equal, and that one of them is at rest; then $\frac{m\,u}{m+n}$, which expresses the velocity after collision, the bodies being considered as unelastic, becomes $\frac{m\,u}{2\,m}$ or
297. $\frac{1}{2}u$. From twice $\frac{1}{2}u$ or u, therefore, we subtract u to obtain the velocity of the impinging body after collision, which is consequently zero. To find the velocity of the impinged body, from twice $\frac{1}{2}u$ or u, we subtract 0, which it had before collision, and we have u for the velocity after collision. Hence we see that the motion of the impinging body passes wholly into the impinged. Accordingly, if several equal elastic bodies be placed in contact with each other in the same straight line, and one of them be made to impinge against the others in the direction of this line; the only effect would be, that the one at the opposite extremity would be driven off with the same velocity. If two are made to impinge at the same time against the others, two would be detached from the other extremity, and so on.

Let us suppose the two bodies to move in the same direction, one of 5 ounces, and with a velocity of 6 feet in a second, and the other of 7 ounces, with a velocity of 2 feet in a second. For the common velocity which they would have after collision, considered as unelastic, we obtain

$$\frac{5 \times 6 + 7 \times 2}{5 + 7} = \tfrac{44}{12} = 3\tfrac{2}{3}.$$

If, therefore, from double this quantity or $7\frac{1}{3}$, we take the velocities before collision, namely, 6 and 2 respectively, we shall have

for the velocity of the impinging body after collision $1\frac{1}{3}$, and for that of the impinged $5\frac{1}{3}$.

If the impinged body, instead of 7 ounces, had a mass of 20 ounces; the velocity, after collision, the bodies being considered as unelastic, would be,

$$\frac{5 \times 6 + 20 \times 2}{5 + 20} = \tfrac{70}{25} = 2\tfrac{4}{5}.$$

If from double this quantity or $5\frac{3}{5}$, we subtract the velocities before collision, namely, 6 and 2 respectively, we shall have,

$$5\tfrac{3}{5} - 6 \quad \text{and} \quad 5\tfrac{3}{5} - 2,$$

that is

$$-\tfrac{2}{5} \quad \text{and} \quad 3\tfrac{3}{5},$$

for the velocities after collision, in which the sign — before $\frac{2}{5}$ indicates that the impinging body would rebound.

If the two bodies are made to move in opposite directions with the same masses and the same velocities, as in the first of the above examples, the velocity after collision, the bodies being considered as unelastic, would be 289.

$$\frac{5 \times 6 - 7 \times 2}{5 + 7} = \frac{30 - 14}{12} = \tfrac{16}{12} = 1\tfrac{1}{3}.$$

If from double this velocity or $2\frac{2}{3}$, we subtract the velocity 6, which the impinging body had before collision, we shall have $-3\frac{1}{3}$ for its velocity after collision; it will rebound, therefore, with a velocity of $3\frac{1}{3}$ feet. As to the impinged body, it will be recollected that to twice $1\frac{1}{3}$ or $2\frac{2}{3}$, the velocity before collision is to be added, which gives $4\frac{2}{3}$ for its velocity after collision. 297.

301. Since, when elastic bodies move in the same direction before collision, the velocities after collision are 297.

$$2\,u' - u \quad \text{and} \quad 2\,u' - v,$$

u' being the velocity which they would have, considered as unelastic; the difference $u - v$ of these two velocities, is the same as the difference of the velocities before collision. This difference is called the relative velocity, and is accordingly the same before and after collision.

When, on the other hand, the bodies move before collision in
397. opposite directions, their velocities after collision are,

$$2\,u' - u \quad \text{and} \quad 2\,u' + v,$$

the difference of which is $u + v$, and this was their relative velocity, or that with which they approached each other before collision. Therefore the velocity with which they separate from each other after collision, is the same as that with which they approach each other before collision; thus, *with respect to clastic bodies the relative velocity is the same before and after collision.*

Of the Motion of Projectiles.

302. By the motion of projectiles, we understand that of bodies, which, being thrown with a certain force, are afterward left to the action of this force and that of gravity. We shall first seek the path that would be described in free space.

Fig.149. From the point A, let a body be thrown in the direction AZ, and with any given velocity. If gravity were out of the question, it would move uniformly in the direction of the straight line AZ. But as gravity acts without interruption, the body will not be in the straight line AZ, except for an instant; instead of AZ, it will describe a curved line ABC of which AZ will be the tan-
Geom. gent at the point A, since AZ is one of the instantaneous direc-
97. tions of the moving body.

303. In order to determine the nature of this curve, let AE be the velocity communicated to the projectile, or the number of feet that it would describe in a second, if it preserved continually this velocity; and at the instant of its leaving the point A, let us suppose this velocity composed of two others, one AD horizontal, and the other AF in a vertical direction. It is evident that the direction of gravity being vertical or perpendicular to AD, its action will not tend either to diminish or increase the velocity AD, and that consequently whatever course the body may take, it will preserve constantly the same velocity parallel to the horizon.

As to the velocity in the direction AF, when the body, in virtue of its constant velocity, parallel to the horizon, shall have

advanced by a quantity equal to AP, it will not have risen to a height PN, equal to that at which it would have arrived, uninfluenced by gravity, but to some lower point M in the same vertical PN; because, its velocity in a vertical direction being directly opposed to that of gravity, the space which it would have described in virtue of this vertical velocity, must be diminished by the space which the action of gravity would have caused the body to describe in the same time.

Accordingly let v denote the velocity communicated in the direction AZ, or the number of feet that the projectile would describe uniformly each second, in virtue of this velocity, and t the time, or number of seconds or parts of a second, employed in passing from A to some point N, we shall have 263.

$$AN = v\,t.$$

Let g be the velocity communicated by gravity in a second, $\frac{1}{2}\,g\,t^2$ will be the space that a heavy body would describe in a number t of seconds. If therefore M be the point where the body will arrive at the expiration of the time t, we shall have

$$NM = \tfrac{1}{2}\,g\,t^2.$$ 275.

Through the point A, draw the vertical AX, and through the point M the straight line MQ parallel to the tangent AZ. Calling AQ, x', and QM, which is equal to AN, y', we shall have

$$x' = \tfrac{1}{2}\,g\,t^2, \text{ and } y' = v\,t.$$

If from this last equation we deduce the value of t, namely,

$$t = \frac{y'}{v},$$

and substitute it in the first, we shall obtain

$$x' = \frac{\frac{1}{2}\,g\,y'^2}{v^2},$$

or

$$\frac{v^2\,x'}{\frac{1}{2}\,g} = y'^2.$$

But $\frac{v^2}{2\,g}$ expresses the height from which a heavy body must fall to acquire the velocity v; hence, if we call this height h, we shall 277.

have $\frac{v^2}{2g} = h$, and consequently $\frac{v^2}{\frac{1}{2}g} = 4\ h$;

therefore,

$$4\ h\ x' = y'^2.$$

We hence infer that each point M of the curve AMC has this property, that the square of the ordinate y' or QM, parallel to the tangent AZ, is equal to the product of the abscissa AQ or x' by
Trig. 176. a constant quantity $4\ h$; therefore the curve AMC is a parabola which has for a diameter the vertical line AX, and for its parameter the quadruple of the height due to the velocity of projection, and of which the angle AQM, made by the ordinates with this
Trig. 182. diameter, is the complement of the angle of projection ZAC. This curve, therefore, is easily constructed, when the velocity of projection and the angle of projection are known.

304. We proceed to examine some of the properties of this curve, considered as the path traced by a projectile; and for this purpose we refer the different points M to the horizontal line AC by drawing PM perpendicular to AC.

We designate AP by x, PM by y, and the angle of projec-
Trig. 30. tion ZAC by a. In the right-angled triangle APN we have

$$\begin{aligned} 1 : AN &:: \sin NAP : PN, \\ &:: \cos NAP : AP; \end{aligned}$$

whence

$$PN = AN \sin NAP = v\,t \sin a,$$

and $\qquad AP$ or $x = v\,t \cos a$.

Also, since $MN = \frac{1}{2}\,g\,t^2$, as we have seen above,

$$PM \text{ or } y = v\,t \sin a - \tfrac{1}{2}\,g\,t^2.$$

Deducing from the former equation the value of t, namely,

$$t = \frac{x}{v \cos a},$$

and substituting it in the latter, we shall have,

$$y = \frac{x \sin a}{\cos a} - \frac{\frac{1}{2}\,g\,x^2}{v^2 \cos a^2},$$

or,

$$\frac{v^2}{\frac{1}{2}g}\, y = \frac{v^2\, x \sin a}{\frac{1}{2} g \cos a} - \frac{x^2}{\cos a^2},$$

or, putting for $\frac{v^2}{\frac{1}{2}g}$ its value $4\,h$, and multiplying both members by $\cos a^2$,

$$4\,h\,y\,\cos a^2 = 4\,h\,x\,\sin a\,\cos a - x^2,$$

which will furnish us with the following properties.

305. As the velocity communicated to the projectile is supposed to be limited to a certain measure, its effect in a vertical direction must be exhausted at the end of a certain time by the action of gravity, so that at a certain point the body will cease to ascend, and thence will commence a downward motion; but, as its horizontal velocity does not change when it has reached its highest point, as B, it will describe the second branch BC of the same curve, and will again meet the horizontal line AC in another point C. Now in order to determine the distance AC, called the *horizontal range* * of the projectile, we have only to suppose $y = 0$. We have, accordingly,

$$4\,h\,x\,\sin a\,\cos a - x^2 \text{ or } x\,(4\,h \sin a \cos a - x) = 0;$$

which gives $x = 0$, and $x = 4\,h \sin a \cos a$. The first value of x indicates the point A; the second is that of AC, which may be determined by producing XA till AK is equal to $4\,h$, and letting fall from the point K upon AZ the perpendicular KL, and from the point L upon AC the perpendicular LC; since we have

$R = 1 : \sin K = \sin a :: AK = 4\,h : AL = 4\,h \sin a$,
and $R = 1 : \sin ALC = \cos a :: AL = 4\,h \sin a : AC = 4\,h \sin a \cos a$.

306. If with the same velocity of projection we would know what angle would give the greatest horizontal range, we take the differential of the value of AC, by regarding a as variable, and put this differential equal to zero; thus Cal. 45.

$$4\,h\,d\,a \cos a^2 - 4\,h\,d\,a \sin a^2 = 0;$$

* Sometimes called also *random* and *amplitude*.

from which we deduce,

$$\sin a^2 = \cos a^2$$

or

$$\frac{\sin a^2}{\cos a^2} = 1,$$

Trig. 8. that is,

$$\text{tang } a^2 = 1,$$

and consequently,

$$\text{tang } a = 1,$$

in other words, the tangent of the angle of projection is in this case equal to radius; accordingly this angle is equal to 45°.
Trig. 24. Therefore, *the greatest horizontal range is obtained, other things being the same, when the angle of projection is* 45°. It is here supposed that

Trig. 20.

$$\sin a = \cos a = \sqrt{\tfrac{1}{2}};$$

this value substituted in the above expression for AC, gives

$$AC = 4\,h\sqrt{\tfrac{1}{2}}\sqrt{\tfrac{1}{2}} = 4\,h \times \tfrac{1}{2} = 2\,h;$$

therefore, *the greatest random is double the height through which a body must fall to acquire the velocity of projection.*

307. If we would know to what height the body ascends, or the highest point B of the curve, we proceed thus; in the equation

$$4\,h\,y\cos a^2 = 4\,h\,x\sin a\cos a - x^2,$$

we put equal to zero, the differential of y, taken by regarding x only as variable, which gives

$$4\,h\,d\,x\sin a\cos a - 2\,x\,d\,x = 0,$$

from which we obtain

$$x = 2\,h\sin a\cos a;$$

205. therefore, since $AC = 4\,h\sin a\cos a$, if we suppose the perpendicular BD, we shall have

$$x \text{ or } AD = 2\,h\sin a\cos a = \tfrac{1}{2}\,AC.$$

Moreover, this value of x being substituted in the equation,

$$4\,h\,y\,\cos a^2 = 4\,h\,x\,\sin a\,\cos a - x^2,$$

gives

$$4\,h\,y\cos a^2 = 8\,h^2\sin a^2\cos a^2 - 4\,h^2\sin a^2\cos a^2,$$

from which we obtain

$$y \text{ or } BD = h\sin a^2.$$

This determines the vertex of the axis, since $d\,y$ being zero at the point B, the tangent at B is parallel to AC, or perpendicular to BD.

308. We propose now to determine the direction AZ, to be given to a projectile in order that it may fall upon a known point M, that is, the inclination that a mortar, for instance, must have to throw a shell upon the known point M. Fig. 150.

The perpendicular MP upon the horizontal line passing through the point A, being drawn, the distance AP, and the angle MAP, are to be considered as known. AP being designated by c, and the angle MAP by e, we shall have

$$\cos e : c :: \sin e : MP = \frac{c\sin e}{\cos e},$$

we have, therefore, for the point M, $x = c$, and

$$y = \frac{c\sin e}{\cos e}.$$

Substituting these values in the equation

$$4\,h\,y\cos a^2 = 4\,h\,x\sin a\cos a - x^2,$$

we obtain,

$$\frac{4\,h\,c\sin e\cos a^2}{\cos e} = 4\,h\,c\sin a\cos a - c^2,$$

or

$$4\,h\sin e\cos a^2 = 4\,h\sin a\cos a\cos e - c\cos e,$$

or

$$4\,h\cos a\,(\sin a\cos e - \sin e\cos a) = c\cos e,$$

that is, Trig. 11.

$$4\,h\cos a\sin(a - e) = c\cos e;$$

or, since

Trig. 27. $\cos a \sin (a - e) = \frac{1}{2} (\sin (a + a - e) - \sin (a - a + e)),$

$$4 h \tfrac{1}{2} (\sin (2 a - e) - \sin e) = c \cos e,$$

and

$$\frac{2 h}{\cos e} \sin (2 a - e) = \frac{2 h \sin e}{\cos e} + c,$$

which may be given by the following construction.

Having raised upon AM the indefinite perpendicular AE; from the middle D of $AK = 4 h$, we erect upon AK the perpendicular DE, cutting AE in some point E, from which as a centre, and with a radius equal to EA, we describe the arc $ANN'K$; having produced PM till it meets this arc in the points N, N', if we draw the lines ANZ, $AN'Z'$, these will be the directions in which, the projectile being thrown with a velocity due to the height h, it will in either case fall upon the point M.

Geom. 209. Indeed it will be readily seen that the angle EAD of the right-angled triangle ADE is equal to MAP. Therefore, since

$AD = 2 h$, $ED = \frac{2 h \sin e}{\cos e}$; and, since $AP = c$, we shall have,

$$ED + AP \text{ or } EI = \frac{2 h \sin e}{\cos e} + c,$$

consequently,

$$\frac{2 h}{\cos e} \sin (2 a - e) = EI.$$

But in the same triangle ADE, $AE = \frac{2 h}{\cos e}$; therefore

$$AE \sin (2 a - e) = EI.$$

Let the arc KNA be produced till it meets, in G, the vertical GE, and from the points N, N', draw the perpendiculars NL, $N'L'$. In the triangle NEL, we have

$$NE : NL, \text{ or } AE : EI :: 1 : \sin NEG;$$

whence,

$$AE \sin NEG = EI;$$

accordingly,

$$\sin (2a - e) = \sin NEG,$$

and

$$2a - e = NEG = NEA + e;$$

consequently,

$$a = \tfrac{1}{2} NEA + e.$$

But because the angle NAM has its vertex in the circumfer- Geom.
ence, and AM is a tangent, NAM is equal to $\frac{1}{2}$ NEA; also the 131.
angle $MAP = e$; whence

$$a = NAM + MAP = NAP;$$

therefore the point N satisfies the question.

The same may be shown with respect to the point N'. Indeed in the triangle $N'EL'$, we have

$$N'E : N'L', \text{ or } AE : EI :: 1 : \sin N'EL',$$
$$:: 1 : \sin N'EG,$$

whence

$$AE \sin N'EG = EI;$$

and, since

$$AE \sin (2a - e) = EI,$$

as above shown, we have

$$\sin (2a - e) = \sin N'EG,$$

and

$$2a - e = N'EG = N'EA + e;$$

therefore,

$$a = \tfrac{1}{2} N'EA + e = NAM + MAP = N'AP.$$

309. Thus with the same force of projection, a projectile may always be made to fall upon the same point M, according to two different directions, provided that AP does not exceed DR. The direction AN' is the most favorable for crushing buildings or other objects with shells. The direction AN is to be preferred, when the purpose is simply to throw down walls and breast-works, and by rebounding to lay waste at a distance. This leads us to speak of *ricochet* firing; but we shall first remark that the equation $x = vt \cos a$, found above, gives a simple ex-

pression for the time employed in passing from A to any point M. We have only to put for x its value c, and for v its value
277. $\sqrt{2gh}$, and we have

$$t = \frac{c}{\cos a \sqrt{2gh}}.$$

Now we have seen how h is determined by experiment, and we know that $g = 32{,}2$ feet.

Trig. 20. When the inclination is 45°, $\cos a$ being $\sqrt{\frac{1}{2}}$ or $\frac{1}{2}\sqrt{2}$, we have

$$t = \frac{c}{\frac{1}{2}\sqrt{2}\sqrt{2gh}} = \frac{c}{\sqrt{gh}};$$

hence, if the point M is in a horizontal line, which gives c equal
306. to the range or to $2h$, we obtain

$$t = \frac{2h}{\sqrt{g}\sqrt{h}} = 2\sqrt{\frac{h}{g}}.$$

This general expression for the time may be made use of in regulating the fusees of bombs. We proceed now to the subject of ricochet firing.

310. By the above term is meant a motion by which a projectile, after meeting with an obstacle, rebounds and commences a new motion similar to the first. The smaller the angle of elevation above the horizon, the greater, other things being the same, is the tendency, upon rebounding, to proceed forward; since the projectile force is exerted almost entirely in a horizontal direction, and much time is required for the resistance of the air and other obstacles to destroy it. If the projectile be of an unelastic substance, and the surface upon which it falls be horizontal and
Fig. 151. unyielding, it would not bound, since upon arriving at C, according to any direction MC, its velocity might be decomposed into two others, of which QC, perpendicular to the surface, would be simply destroyed, the rebounding in other cases being caused entirely by the elasticity, so that the other part PC remains (no account being taken of friction and the resistance of the air), and the body would move along CZ.

311. But if at the point C, where the body meets the surface Fig. 152. there be a mound or eminence CE, the motion according to MC being decomposed into two others, one according to QC perpendicular to the surface CE, and the other PC in the direction of this surface, the body would proceed according to this latter, describing the line PE, and might, after leaving the point E, describe just such a curve as it would have described, if it had been projected from E, according to CE, with the same velocity; so that it would elevate itself to a certain point, and then return to the surface in some other point I, when the motion under similar circumstances might be again renewed.

312. A ricochet motion, therefore, depends upon the position of the obstacle against which the body in question strikes. But if the obstacle be flexible or yielding like the earth, water, &c., this motion may take place even when the surface is perfectly horizontal. Indeed, by the vertical velocity QC, the body tends to bury Fig. 153. itself, and does bury itself more or less, according to the nature of the obstacle; while with the velocity PC it ploughs the earth, and forms a furrow, the depth of which increases till the vertical velocity QC is destroyed. Then by the remaining velocity in a horizontal direction, it drives before it the matter which lies in its way, and in working for itself a passage, it inclines in the direction from which it experiences the least resistance, and the surface of the furrow becomes, with respect to the body, what CE was in the last case. Now as the remaining projectile Fig. 152. force, other things being the same, is so much the greater according as the depth of the furrow is less, and as this depth depends upon the vertical velocity QC which will be so much the less, according as the angle MCP, or the angle of projection RAZ, is less, it will be seen how the smallness of the angle of projection is favorable to this sort of motion.

313. The figure of the body also is of great importance. If, for example, the question related to a motion upon water, and the body were of a spherical shape, the velocity MC must be such that the vertical velocity QC may be destroyed before the vertical diameter of the body is entirely immersed, since, when the body is once covered, the resistance of the water would act equally in every direction, and there would be nothing to change

its direction, except gravity, the tendency of which would be to prevent a ricochet.

314. As this immersion, however, takes place gradually, it will be seen that the motion of the centre must be in a curved line;
Fig. 154. since while any part of the body remains above the surface, the direction in which the resistance acts is changing continually. If, for instance, when the centre C, after having described any line PC, tends to move according to the prolongation CI, of this line, we imagine two tangents BR, DS, parallel to this direction, it is evident that the part BVL only would be exposed to this resistance; and that if the body is spherical, the resultant CK of all the resistances exerted upon the different points of BVL would have a direction tending to elevate the body above CI; so that the parallelogram $CIEK$ being formed, CE will be the course which the body would take instead of CI, no allowance being made for gravity.

315. Finally, if the body and the obstacle are flexible and elastic, this circumstance will further contribute to a ricochet motion. We take a very simple case, as an example; let the
Fig. 155. body only be considered as flexible and elastic, and let this elasticity be perfect; the body being supposed at the same time to be destitute of gravity. At the instant in which the body, projected according to AC, comes to touch the surface, its velocity is decomposed into a horizontal velocity which would remain always the same, if there were no friction, and no resistance on the part of the medium in which the body moves. As to the perpendicular or vertical velocity PC, it compresses the body, and being destroyed gradually, while the horizontal velocity continues, it is evident that the centre C approaches the plane HZ by degrees, which go on decreasing, while the rate at which it advances parallel to HZ, remains the same. Consequently, if at each instant we imagine a parallelogram having its horizontal sides to its vertical, as the horizontal velocity is to the velocity that remains in a vertical direction, the diagonal of this parallelogram, which must mark the course of the centre each instant, will be different and differently situated each instant, so that the centre C will approach HZ in a curve, while the compression is going on. When the compression has ceased, the centre C will

be carried for an instant in the direction of a tangent parallel to HZ; after which, the recoil taking place, the body recovers by degrees the velocity by which it tends to depart from the plane after the same manner in which the velocity was destroyed by the compression during its approach to the plane, and it will describe the second part RO of the curve perfectly similar to RC. Lastly, when it shall have arrived at the point O, distant from the plane HZ by a quantity equal to the radius IC, it will move according to the tangent OT, situated like AC; that is, the oblique collision of a body against an inflexible and unelastic plane (friction being out of the question) takes place in such a manner as to make the angle of reflection equal to the angle of incidence, these angles having for their measure the inclination to a horizontal plane of the tangents at the extremities C, O, of the curve described by the centre of the body during its compression and subsequent recoil.

316. If BD be the direction in which a body is thrown, regard being had to gravity, this body will describe the portion DC of a parabola of which BD is the tangent, until it touches the plane, then, when the compression has ceased, it will describe another portion SO of a parabola equal to the first and placed in the same manner.

317. Friction, moreover, contributes to the kind of motion under consideration, since it occasions a rotation in the body that aids it in rising above obstacles, as we have already seen. 234.

318. We conclude what we have to say on the subject of projectiles moving in an unresisting medium, with observing that, since gravity draws a body downward from the direction given it by the projectile force, when we take aim at an object in shooting or in throwing any body, we should direct the sight above this object, and so much the more above it, according as it is more distant, and according also to the feebleness of the force employed. It is on this account that in fire-arms the line of sight makes an angle with the axis of the piece, so that these lines produced would meet at a point beyond the muzzle toward the mark. The projectile, ball, or bullet, propelled in the direction of the axis, commences its motion in a direction making a greater angle with the horizon than that made by the line of sight; so that the

precaution is the same as if we had taken aim in the direction of the axis but at a point above the object.

319. We remark further, that there are cases in which, although we have given no impulse to a body, and seem to abandon it to gravity alone, yet this body describes a curved line common to all projectiles. A body, for example, which is suffered to fall from the mast-head of a vessel under sail, really describes a curved line. If we attend to the point of the deck where it strikes, we shall find it just as far from the mast, other things being the same, as the point from which it started, so that the body describes a line parallel to the mast; but with respect to a spectator at rest, it has actually described a parabola (the resistance of the air not being considered), for, at the instant it was dropped, it must have had the same velocity with the vessel; the case is therefore precisely the same, as if, the vessel being stationary, we had thrown it with a velocity equal to that of the vessel, and in the same direction. It will be seen, also, at the same time, why it describes with respect to the mast a straight line parallel to this mast; it is because they both move with the same velocity, and in the same direction; considered horizontally, therefore, they must preserve the same distance from each other.

320. In the foregoing theory, we have taken it for granted; (1.) that the force of gravity is the same throughout the whole range of the projectile. (2.) That it acts in lines parallel to each other. (3.) That there is no resisting medium. The two first suppositions, although not strictly conformable to fact, are attended with no material error in practical gunnery, and those arts to which this theory is subservient. But the third is of essential importance to the truth of the results we have obtained. We can readily put the theory to the test of actual experiment.

The initial velocity of a cannon-ball, for instance, may be obtained with considerable accuracy, by either of the following methods.

321. (1.) Let the cannon together with the carriage and other weight if necessary, be suspended like a pendulum, so as to move freely in the direction opposite to that in which the

ball is to be discharged.* Upon the explosion taking place, the centre of gravity will remain unchanged, that is, the quantities 134. of motion in opposite directions will be equal; consequently, if the motion of the gun, &c., be made so slow by means of the attached weight, as to admit of its velocity being taken by actual observation, the velocity of the ball will be as much greater as its mass is less. Knowing the mass of each, we should use the following proportion; as the mass or weight of the ball to that of the gun, carriage, &c., so is the velocity of the latter to that of the former.

322. (2.) The ball may be discharged into a large block of wood suspended so as to move freely after the manner of a pendulum,* and, the velocity being observed as before, we then say as the mass of the ball to that of the pendulous body, so is the velocity of the latter to that of the former. This latter method is adapted to finding the velocity at different distances from the cannon.

It is thus found that the velocity of a cannon-ball varies according to the quantity and quality of the powder, the size of the ball, the length of the piece, &c. At the commencement of the motion, it is ordinarily between 800 and 1600 feet in a second.

323. With a velocity equal to 800 feet in a second, the angle of projection being 45°, for instance, the horizontal range, greatest 306,307, &c. elevation, &c., are readily determined by our formulas.

We first find the height h through which a body must fall to acquire the velocity of projection 800 feet, and double this height will be the horizontal range required. Now to acquire a velocity of 800 feet in a second, a body must fall through a space equal

to $\frac{(800)^2}{64,4}$ feet. 277.

800 ft....log....2,90309
2

5,80618
64,4....log....1,80889

$h = 9937,75$ . . . 3,99729
2

Range = 19875,5 = 3,7 miles.

* It will be seen hereafter at what point in the pendulum the impulse must be applied in order that no part of it may be expended against the supports from which the pendulum is suspended.

The greatest elevation is equal to h multiplied by the sine square of the angle of projection, that is, equal to h (sin $45^\circ)^2$.

$$\begin{array}{rr} h = 9937{,}75\text{ft. log} & 3{,}99729 \\ 45^\circ \text{ log sin} & 9{,}84949 \\ & 9{,}84949 \\ \hline \end{array}$$

Greatest elevation $= 4969$ feet $\qquad 3{,}69627$

4969 wants only 311 feet of a mile.

Moreover, according to the case supposed, we have $2\sqrt{\dfrac{h}{g}}$ as the expression for t the time of flight.

$$\begin{array}{lr} h = 9937{,}75\ldots\text{log}\ldots & 3{,}99729 \\ g = 32{,}2 \qquad \text{log} & 1{,}50786 \\ \hline & 2)2{,}48943 \\ \hline 17'',57 & 1{,}24472 \\ 2 & \\ \hline t = 35,\ 14 & \end{array}$$

On the supposition of a velocity of 1600 feet in a second, the angle of projection being the same, we should have for the horizontal range 79503 feet or 15 miles, for the greatest elevation 3,7 miles, and for the time of flight 3 minutes and 38 seconds. So great, however, is the resistance of the air, that a cannon-ball, under the most favorable circumstances, is seldom known to have a range exceeding 3 miles; the path described is not strictly a parabola or any known curve; its vertex is not in the middle, but more remote from the point of projection than from the other extremity; and the path through which the body descends is less curved than that through which it ascends. This resistance increases faster than the velocity; so that in the slower motions, there is a nearer approach to the foregoing theory, than in those which are more rapid, as is apparent to the eye in the spouting of water, and more especially of mercury, from the side of a vessel. To treat of this resistance, and to estimate its effects, belongs to that branch of our subject which has for its object the motion of fluids and that of bodies immersed in them.

Of the Motion of heavy Bodies down inclined Planes.

324. A heavy body left to itself upon a plane surface *KLHI*, inclined to a horizontal surface *PIHN*, cannot yield entirely to its gravity. A part of the force derived from this cause is employed in pressing the plane, and the other serves to bear it along the plane. It is necessary, therefore, to decompose its gravity into two forces, one of which produces the pressure upon the plane, and the other the motion along this plane. Fig. 156. 37.

325. Let *G* be the centre of gravity of the body *m*, or the point in which all its action may be considered as united. Let *GB* be the space through which it would fall in an instant, if it were free. Let *GC* be drawn perdendicular to the plane; and suppose a plane to pass through *GB*, *GC*, this plane will be perpendicular to the two planes *KLHI*, *IPNH*, since it passes through Geom. 351.
the straight lines perpendicular to these planes. If, therefore, we conceive *DE*, *EF*, to be the intersections of this plane with *KLHI*, *IPNH*; *DE*, *EF* will be perpendicular to the common intersec- Geom.
tion *HI* of these two planes. 355.

Draw *GA* parallel to *DE*, and construct the parallelogram *GABC* of which *GB* is the diagonal, and *GA*, *GC*, the sides. We may suppose that gravity, instead of urging the body according to *GB*, urges it at the same time according to *GC* with the velocity *GC*, and according to *GA* with the velocity *GA*. Now it is evident that *GC*, being perpendicular to the plane, cannot but be destroyed, if the point *O* where it meets the plane is at the same time a point common to the plane and the body *m*.

As to the force *GA*, since it tends neither to approach toward, nor to recede from the plane, it cannot but have its full effect. *GA*, therefore, represents the velocity with which the body tends to move, and with which it would move in the first instant.

As the force *GA* is in the plane of the two right lines *GB*, *GC*, it is in the plane *DEF*. We can therefore leave out of consideration the extent of the two planes *KLHI*, *IPNH*, and employ only the plane *DEF* represented in figure 157, so that the body may be supposed to move in the right line *DE*.

326. Since the force GA passes through the centre of gravity G of the body m, it must distribute itself equally to all parts of this
116. body. Therefore, so long as friction is supposed to have no influence, the body can have no motion except that of sliding along the plane, that is, it can have no tendency to roll, whatever may be its figure, provided the perpendicular GB meets the plane in a point that belongs at the same time to the surface of the body. This would not be the case, however, as we have seen, if the per-
196. pendicular did not meet the base of the body, or the surface by which it rests upon the plane. The influence of friction, moreover, tends to produce a rolling motion.

327. Since the body m must describe GA in the same time in which it would describe GB by the free action of gravity, if we conceive that at the end of the first instant, gravity acts anew; as it communicates in equal instants equal degrees of velocity, by supposing for the second degree of velocity communicated in a vertical direction, a decomposition similar to that above made for the first instant, it is evident that the second parallelogram will be equal in all respects to the first. We accordingly infer, in like manner, that the force perpendicular to the plane will be destroyed, and the force parallel to the plane, and equal to GA, will be added to GA. By reasoning in the same manner for the following instants, we should conclude that the velocity along the inclined plane is accelerated by equal degrees; in other words, that *the motion of heavy bodies down an inclined plane is a motion uniformly accelerated.* Hence all that has been said upon the subject of motion uniformly accelerated, is strictly applicable to the motion that takes place down inclined planes. Consequently in this latter case, as well as in the former, the velocities are as the times, the spaces described are as the squares of the times, or as
264, &c. the squares of the acquired velocities, &c.

328. Therefore, in order to determine the motion that takes place upon a plane of a known inclination, we have only to find the ratio of the accelerating force to gravity, that is, the ratio of GA to GB. Now GA, GB, being parallel respectively to DE, DF, the angle AGB is equal to EDF, and, the angle A being a right angle as well as the angle F, the two triangles AGB, EDF, are similar; whence,

$$DE : DF :: GB : GA;$$

that is, *the length of the inclined plane is to its height, as the velocity which gravity communicates to a free body, is to that with which it urges the body along the inclined plane.*

329. Now as gravity gives to a free body, in a second of time, a velocity by which a space of 32,2 feet are described uniformly in a second, it will be easy to determine the velocity acquired 273. by a body in the first second of its descent down an inclined plane. If, for example, the length of the plane is double the height, the velocity acquired along the plane during the first second, will be half of 32,2 feet; that is, at the end of the first second, if gravity ceased to act, the body would pass over 16,1 feet in a second.

Having thus determined the velocity for the first second, we shall have the velocity after any proposed number of seconds, by multiplying this by the number of seconds; also the space is found by multiplying this first velocity by half the square of 267. the number of seconds. In short, it would be easy to determine all the other circumstances of the motion in question, by articles 267, &c. We hence deduce the following propositions.

330. If two heavy bodies, setting out at the same time from the point D, descend, one along the plane DE, and the other in Fig. 153. the direction of the perpendicular DF, and we would know in what part of the plane DE the first would be, when the second had arrived at any given point A, we have only to let fall from the point A upon DE the perpendicular AB; and the point B will be the place sought. Indeed if we represent by g the velocity that gravity communicates to a free body in one second, by calling t the time employed in describing DA, we shall have

$$DA = \tfrac{1}{2} g t^2, \qquad 267.$$

on the other hand the velocity acquired in a second by the body that descends along the plane DE, is $\frac{g \times DF}{DE}$; accordingly by 328. calling t' the time employed in descending from D to B, we shall have

$$DB = \frac{g \times DF}{DE} \times \tfrac{1}{2} t'^2;$$

whence,

$$DA : DB :: \tfrac{1}{2} g\, t^2 : \frac{g \times DF}{DE} \times \tfrac{1}{2} t'^2,$$

$$:: DE \times t^2 : DF \times t'^2.$$

But by similar triangles,

Geom. 213.

$$DA : DB :: DE : DF;$$

consequently,

$$DE : DF :: DE \times t^2 : DF \times t'^2;$$

therefore,

$$t'^2 = t^2 \quad \text{or} \quad t' = t.$$

Fig. 159. 331. Hence, if DG be a third plane described by a third body setting out from D at the same time with the other two, by drawing from the point A the perpendicular AC, A, B, C, are the three points at which the three would arrive in the same time.

332. If upon DA as a diameter, we describe a semicircumference, it will pass through the points C and B, since the angles at C and B are right angles. Consequently, the chords DC, DB, and the vertical diameter DA, are all described in the same time; and as this does not depend upon the length or inclination of the chords, we may draw the general conclusion, that *the time employed by a body in falling through any chord of a circle, drawn from the extremity of a vertical diameter, is the same as that employed in falling through this vertical diameter.*

Geom. 128.

333. We have seen that g being the velocity communicated to a free body in one second, $\frac{g \times DF}{DE}$ is that given in the same time to a body that descends along DE. Let t, t', be the times employed in describing DF and DE respectively; we shall have

$$DF = \tfrac{1}{2} g\, t^2,\ DE = \frac{g \times DF}{DE} \times \tfrac{1}{2} t'^2;$$

whence,

$$DF : DE :: \tfrac{1}{2} g\, t^2 : \frac{g \times DF}{DE} \times \tfrac{1}{2} t'^2,$$

which gives, by multiplying the extremes and means and reducing,

$$\frac{\overline{DF}^2}{DE} \times t'^2 = DE \times t^2,$$

or

$$\overline{DF}^2 \times t'^2 = \overline{DE}^2 \times t^2,$$

or, taking the square root of each member,

$$DF \times t' = DE \times t;$$

in other words,

$$t : t' :: DF : DE.$$

In like manner, if t'' represent the time employed in describing DG; we shall have

$$t : t'' :: DF : DG;$$

whence,

$$t' : t'' :: DE : DG;$$

that is, *the times employed in describing different planes of the same height, are to each other as the lengths of these planes.*

334. The velocity of the body which descends along DF, is $g\,t$ at the expiration of the time t. For a similar reason, the 267.
velocity of the body that descends along DE, is $\frac{g \times DF}{DE} \times t'$ at the expiration of the time t'. Accordingly, if we call u, v, the velocities acquired by the two bodies respectively upon arriving at the points F, E, we shall have,

$$u : v :: g\,t : \frac{g \times DF}{DE} \times t',$$

whence,

$$v\,g\,t = u\,g \times \frac{DF}{DE} \times t'.$$

But, as we have just seen, 322.

$$t : t' :: DF : DE,$$

which gives

$$t = \frac{DF \times t'}{DE};$$

substituting for t this value in the above equation, we shall have,

$$v = u.$$

Therefore, *if several bodies descend along planes differently inclined, but of the same height, they will have the same velocity upon arriving at the same horizontal line.*

Of Motion along curved Surfaces.

335. If a body without gravity and without elasticity, describes, in virtue of a primitive impulse, the successive sides AB,
Fig.160. BC, &c., of any polygon, upon meeting each side it will lose a part of its velocity, which may be determined in the following manner.

Let us suppose that the body moves from A toward B, and that when it is at B, its velocity is such as in a determinate time, one second for example, would cause it to describe, if it were free, the line BF in AB produced. Having erected upon BC from the point B, the perpendicular BE, we imagine the rectangular parallelogram $BDFE$, of which BF is the diagonal, and the sides of which are in the direction of BC and BE. Instead of the velocity BF, we may suppose that the body has at the same time the two velocities BD, BE; and as the side BC prevents its obeying the velocity BE, it is manifest that its velocity is reduced to BD.

If from the point B, as a centre, and with a radius BF, we describe the arc FI, DI, which is the difference between BF and BD, will accordingly be the velocity lost. Now DI is the versed
Trig. 5. sine of the arc FI, or of the angle FBC, made by the two contiguous sides AB, BC. Therefore so long as these two sides make a finite angle, the body will lose a finite part of its velocity upon meeting each of the sides.

336. But if the angle formed by the two sides is infinitely small, the velocity lost will not only not be a finite quantity, but

it will not be an infinitely small quantity of the first order, it will only be an infinitely small quantity of the second order. In es- Cal. 4. tablishing this, the question reduces itself to showing that the versed sine of an infinitely small arc is an infinitely small quantity of the second order; and this may be done thus. CD being Fig. 161. any arc of a circle, and BD a perpendicular upon the diameter AC, we have Geom. 215.

$$AB : BD :: BD : BC;$$

hence, if CD, (and for a stronger reason BD) be infinitely small, BC the versed sine of CD, will be infinitely smaller than BD, since it is contained in BD as many times as BD is contained in the infinitely greater quantity AB. Therefore BC is infinitely small of the second order.

337. Accordingly, *if a body without gravity move along the* Fig. 162. *curved surface ABC, it will have throughout the same velocity.* For by considering this curve as a polygon of an infinite number of sides, since the sides make angles infinitely small with each other, the loss of velocity at the meeting of each two adjacent sides is an infinitely small quantity of the second order with respect to the original velocity. Consequently the sum of the velocities lost in passing over an infinite number of sides, that is, in passing over any arc ABC, can only form an infinitely small quantity of the first order. Therefore the velocity is not affected by this circumstance. Cal. 4.

338. We come now to the motion of heavy bodies along curved surfaces. We shall consider for the present only that which takes place in a vertical plane.

339. Accordingly, let AMB be a section of a curved surface, Fig. 163. made by a vertical plane, and the path described by a body along this surface. Let us consider this curve as a polygon of an infinite number of sides, and let us suppose that the body has just described the small side LM. As its meeting with the side MN cannot occasion any loss of velocity; it will describe MN 336. with the velocity which it had in M, gravity being supposed no longer to act upon it. But the force of gravity being exerted according to the vertical MO, urges the body anew as it would urge one upon a plane of the same inclination. Consequently,

if we imagine the velocity MO, which gravity tends to give in an instant, decomposed into two parts, one MD perpendicular to MN, and the other DO or ME directed according to MN; we shall see that it is by virtue of this last that the velocity of the body will be accelerated. Now by letting fall the perpendicular RN, and comparing the similar triangles MOE, MNR, we shall have,

$$MN : NR :: MO : ME = \frac{NR \times MO}{MN}.$$

Let us suppose that the different points of the curve AB are referred to the vertical axis BZ. If we call

$$BP,\ x; \quad PM,\ y; \quad \text{and the arc } BM,\ s;$$

we shall have

$$PQ \text{ or } RN = -\,d\,x; \text{ and } MN = -\,d\,s.$$

We give the sign — to these quantities, because x and s go on diminishing while the time t increases. Let g be the velocity which gravity gives to a free body in a second; $g\,d\,t$ will be that
267. which it would give in the instant $d\,t$. We shall therefore have the velocity represented by MO as follows, namely, $MO = g\,d\,t$.

Calling v the velocity which the body has when it arrives at M; $d\,v$ will denote the augmentation received during the time $d\,t$; thus,

$$ME = d\,v.$$

Substituting the values above obtained in the equation

$$ME = \frac{NR \times MO}{MN},$$

we shall have,

$$d\,v = \frac{-\,d\,x}{-\,d\,s} \times g\,d\,t = \frac{d\,x}{d\,s} \times g\,d\,t.$$

But, by article 280, $d\,s = v\,d\,t$, or $d\,t = \frac{d\,s}{v}$, or, s being considered as decreasing while t increases, $d\,t = \frac{-\,d\,s}{v}$; whence, by substitution,

$$d\,v = \frac{d\,x}{d\,s} \times g \times \frac{-d\,s}{v} = -\frac{g\,d\,x}{v},$$

or

$$v\,d\,v = -g\,d\,x.$$

The integral of this equation is, Cal. 83.

$$\frac{v^2}{2} = C - g\,x,$$

whence,

$$v^2 = 2\,C - 2\,g\,x.$$

In order to determine the constant C, let us suppose that the point A from which the body begins to fall, is elevated above a horizontal line passing through B by a quantity $BZ = h$. It is necessary, therefore, when v is zero, that x should be equal to h; accordingly we have

$$0 = 2\,C - 2\,g\,h,$$

and consequently

$$2\,C = 2\,g\,h, \text{ or } C = g\,h;$$

whence, by substitution,

$$\begin{aligned} v^2 = 2\,g\,h - 2\,g\,x &= 2\,g\,(h - x), \\ &= 2\,g \times ZP. \end{aligned}$$

Now, if a heavy body fall through the space ZP, the square of the velocity which it will have upon arriving at P, will be

$$2\,g \times ZP. \qquad 277.$$

Therefore, *when a body descends along any curved line, it has, at any point whatever, the velocity which it would have acquired by falling freely through a space of the same perpendicular elevation.*

Thus the velocity which a body successively acquires by its gravity in descending along the concavity of a curved line, is altogether independent of the nature of this curve.

340. Hence, if the body, after having arrived at the lowest point B (the tangent to which I supposed to be horizontal), meets the concavity of the same or of any other curve, touching the first in B, it will rise upon this last to a height equal to that from which it descended.

Indeed, let us suppose that the body is actually in B, or that $x = 0$; its velocity will be such, that we shall have

277. $$v^2 = 2\,g\,h\,, \text{ or } u^2 = 2\,g\,h,$$

by calling this velocity u to distinguish it from the other. Let us imagine that with this velocity it ascends along any curve BM'; we shall find by the same reasoning as that above pursued, that its velocity in any point M', is determined by the equation,

$$-\,d\,v' = \frac{d\,x}{d\,s'} \times g\,d\,t,$$

by calling v' the velocity in this case, and s' the arc BM', and observing that v' diminishes according as t', s', and x increase

280. respectively. Consequently, putting for $d\,t$ its value $\frac{d\,s}{v'}$, we shall have

$$-\,d\,v' = \frac{g\,d\,x}{v'} \text{ or } v'\,d\,v' = -\,g\,d\,x\,;$$

and by integrating,

$$v'^2 = 2\,C - 2\,g\,x.$$

But, when $x = 0$, the velocity v' is u; accordingly,

$$u^2 = 2\,C - 0,$$

and since

$$u^2 = 2\,g\,h,$$

we have

$$v'^2 = 2\,g\,h - 2\,g\,x.$$

Now when the body ceases to ascend, $v' = 0$, which gives

$$0 = 2\,g\,h - 2\,g\,x\,;$$

whence we deduce

$$x = h.$$

Therefore the point at which the body will have arrived in any curve BA', will be at the same height as the point A.

341. As to the time employed in describing any arc AM or AB of the curve; since $d\,t = \frac{-d\,s}{v}$, substituting for v its 280.
equal $\sqrt{2\,g\,h - 2\,g\,x}$, we have,

$$d\,t = \frac{-d\,s}{\sqrt{2\,g\,h - 2\,g\,x}},$$

so that it would be necessary by means of the equation of the curve to find the value of $d\,s$ in x and $d\,x$, and having substituted it in the expression for $d\,t$, we should have that of t by integrating.

Of the Motion of Oscillation.

342. We have seen that a heavy body having descended through any arc of a curve AB must, setting aside the resistance of the air and friction, ascend again to the same height in a curve BA' which has at the point B the same horizontal tangent with BA. Accordingly, this body in returning would describe in a
contrary direction the whole extent of the curve $A'BA$; and thus 340.
would continue to move backward and forward without end. This kind of motion is called *oscillation.* We have seen what is in general necessary to determine the duration of each oscillation which must evidently be double the time employed in describing the arc AB, if BA' is the same as AB.

When the curve through which the body descends is circular, and the oscillations take place through small arcs only, they have this remarkable and important property, that their duration
is not sensibly affected by the extent of the arc AB; so that the Fig.164.
arc AB being small, as four or five degrees only at the most, the body will always arrive at B in the same time very nearly, whether it set out from the point A, or from some other point O, taken between A and B.

Thus, retaining the denominations used above, and designating by a the radius BC of the circle BAD, we shall have, by
the nature of the circle, Trig. 101.

$$y^2 = 2\,a\,x - x^2 \text{ or } y = \sqrt{2\,a\,x - x^2};$$

from which the value of $M\,m$, or $d\,s$ or $\sqrt{d\,x^2 + d\,y^2}$ is readily found to be as follows, namely,

Cal. 78.
$$d\,s = \frac{a\,d\,x}{\sqrt{2\,a\,x - x^2}}.$$

But since the arc BM is small, x is small with respect to a, and x^2 may be neglected when taken in connexion with $2\,a\,x$ without material error, which leaves

$$d\,s = \frac{a\,d\,x}{\sqrt{2\,a\,x}}.$$

341. Substituting this value of $d\,s$ in the expression for $d\,t$, we shall have

$$d\,t = \frac{-\,a\,d\,x}{\sqrt{2\,a\,x}\,\sqrt{2\,g\,h - 2\,g\,x}} = \frac{-\,a\,d\,x}{\sqrt{4}\,\sqrt{a}\,\sqrt{x}\,\sqrt{g}\,\sqrt{h - x}}$$

$$= \frac{-\,\frac{1}{2}\,a\,d\,x}{\sqrt{a}\,\sqrt{g}\,\sqrt{h\,x - x^2}},$$

or, since

$$\frac{a}{\sqrt{a}} = \sqrt{a}, \text{ and } \frac{\sqrt{a}}{\sqrt{g}} = \sqrt{\frac{a}{g}},$$

$$d\,t = \sqrt{\frac{a}{g}} \times \frac{-\,\frac{1}{2}\,d\,x}{\sqrt{h\,x - x^2}}.$$

Now as $\frac{a\,d\,x}{\sqrt{2\,a\,x - x^2}}$ expresses the element of an arc of a circle of which the diameter is $2\,a$ and the abscissa x; so, in like manner, $\frac{\frac{1}{2}\,h\,d\,x}{\sqrt{h\,x - x^2}}$ expresses the element of an arc of a circle whose diameter is h and abscissa x. But the line BZ being h, if upon BZ as a diameter we describe the semicircle $BM'Z$, $M'\,m'$ will be this element; so that we shall have

$$\frac{\frac{1}{2}\,h\,d\,x}{\sqrt{h\,x - x^2}} = M'\,m' = d\,(BM');$$

whence

$$\frac{\frac{1}{2} d x}{\sqrt{h x - x^2}} = \frac{d (BM')}{h}.$$

Substituting this value in the expression for $d t$, we have

$$d t = - \sqrt{\frac{a}{g}} \times \frac{d (BM')}{h},$$

and by integrating

$$t = C - \sqrt{\frac{a}{g}} \times \frac{BM'}{h}.$$

We have therefore only to determine the constant quantity C; and it will be seen, that when $t = 0$, that is, when the body sets out from the point A, the arc BM' becomes the semicircumference $BM'Z$; accordingly,

$$0 = C - \sqrt{\frac{a}{g}} \times \frac{BM'Z}{h},$$

whence

$$C = \sqrt{\frac{a}{g}} \times \frac{BM'Z}{h};$$

therefore

$$t = \sqrt{\frac{a}{g}} \times \frac{BM'Z}{h} - \sqrt{\frac{a}{g}} \times \frac{BM'}{h}$$

$$= \sqrt{\frac{a}{g}} \times \frac{ZM'}{h}.$$

We have thus an expression for the time employed in describing any arc AM, the time being supposed to be reckoned in seconds. But when the arc AM becomes AB, that is, at the end of a semi-oscillation, the arc ZM' becomes $ZM'B$; consequently, by calling the duration of a semioscillation $\frac{1}{2} t'$, we shall have

$$\frac{1}{2} t' = \sqrt{\frac{a}{g}} \times \frac{ZM'B}{h},$$

or,

$$t' = \sqrt{\frac{a}{g}} \times \frac{2\,ZM'B}{h}.$$

Geom. 291. Now, π being the circumference of a circle whose diameter is 1,

$$1 : \pi :: h : 2\,ZM'B;$$

whence

$$\frac{2\,ZM'B}{h} = \pi,$$

and consequently

$$t' = \sqrt{\frac{a}{g}} \times \pi,$$

or

$$= \pi \sqrt{\frac{a}{g}}.$$

We have thus an expression for the duration of an entire oscillation; and as this quantity does not contain h, or the height from which the body falls, and which determines the extent of the path described AB, it follows that the time t' does not sensibly depend upon the extent of the arc, so long as this arc is very small. Therefore, *the oscillations which take place in small arcs of a circle are sensibly isocronous or of the same duration.*

343. This property belongs to the small arcs of all curves in which the radius of the evolute at the lowest point is not zero; since the arcs are confounded with those of the circle by which Cal. 79. their curvature is measured.

If we would know the error liable to be committed by taking this value of t' for the duration of a semioscillation in a circle, we proceed thus;

Taking the value found above for $d\,s$, namely,

$$d\,s = \frac{a\,d\,x}{\sqrt{2\,a\,x - x^2}},$$

Cal. *Note* 2. we reduce it to a series, retaining the three first terms only, which are abundantly sufficient for our purpose, and we shall have

$$d\,s = \frac{a\,d\,x}{\sqrt{2\,a\,x}}\left(1 + \frac{x}{4\,a} + \frac{3\,x^2}{32\,a^2}\right);$$

by substituting this value in the expression for $d\,t$, namely,

$$\frac{-\,d\,s}{\sqrt{2\,g\,h - 2\,g\,x}}$$

and reducing, we obtain

$$d\,t = -\,\tfrac{1}{2}\sqrt{\frac{a}{g}}\,\frac{\left(x^{-\frac{1}{2}}\,d\,x + \dfrac{x^{\frac{1}{2}}\,d\,x}{4\,a} + \dfrac{3\,x^{\frac{3}{2}}\,d\,x}{32\,a^2}\right)}{\sqrt{h - x}}.$$

As we know already the integral of the first term, we shall confine ourselves to finding that of the two last. Representing it by $\frac{1}{2}\,t''$, we shall have

$$\tfrac{1}{2}\,d\,t'' = -\,\tfrac{1}{2}\sqrt{\frac{a}{g}}\left(\frac{x^{\frac{1}{2}}\,d\,x}{4\,a} + \frac{3\,x^{\frac{3}{2}}\,d\,x}{32\,a^2}\right)(h - x)^{-\frac{1}{2}}.$$

To obtain the integral of this equation, we have recourse to the method laid down in the *Calculus*, articles 128, &c., and put $\frac{1}{2}\,t''$ equal to the following expression, namely,

$$\tfrac{1}{2}\,t'' = -\,\tfrac{1}{2}\sqrt{\frac{a}{g}}\,(A\,x^{\frac{3}{2}} + B\,x^{\frac{1}{2}})\,(h - x)^{\frac{1}{2}}$$

$$-\,\tfrac{1}{2}\sqrt{\frac{a}{g}}\,C\int d\,x\,x^{-\frac{1}{2}}\,(h - x)^{-\frac{1}{2}} + D.$$

The co-efficients are then determined as follows, namely, Cal. 129.

$$A = -\,\frac{3}{64\,a^2},\; B = -\,\frac{1}{4\,a} - \frac{9\,h}{128\,a^2},\; C = \frac{h}{8\,a} + \frac{9\,h^2}{256\,a^2}.$$

Now the integral of $x^{-\frac{1}{2}}\,d\,x\,(h - x)^{-\frac{1}{2}}$, or of

$$\frac{d\,x}{\sqrt{h\,x - x^2}} \text{ or of } \frac{1}{\frac{1}{2}\,h} \times \frac{\frac{1}{2}\,h\,d\,x}{\sqrt{h\,x - x^2}},$$

is

$$\frac{1}{\frac{1}{2}\,h} \times \text{arc } BM';$$

therefore the whole integral is

$$\tfrac{1}{2}\, t'' = -\tfrac{1}{2}\sqrt{\frac{a}{g}}\left(-\frac{3\,x^{\frac{3}{2}}}{64\,a^2} - \frac{x^{\frac{1}{2}}}{4\,a} - \frac{9\,h\,x^{\frac{1}{2}}}{128\,a^2}\right)(h-x)^{\frac{1}{2}}$$

$$-\sqrt{\frac{a}{g}} \times \frac{BM'}{h} \times \left(\frac{h}{8\,a} + \frac{9\,h^2}{256\,a^2}\right) + D.$$

To determine the constant D, we observe that when $x = h$, t'' must be equal to 0, and that the arc BM' becomes $BM'Z$; in this case, therefore, we have,

$$0 = -\sqrt{\frac{a}{g}} \times \frac{BM'Z}{h} \times \left(\frac{h}{8\,a} + \frac{9\,h^2}{256\,a^2}\right) + D.$$

Substituting the value of D obtained from this equation in the expression for $\frac{1}{2}\, t''$, making $x = h$ in order to have the entire integral, and observing that BM' becomes then zero, the result will be

$$\tfrac{1}{2}\, t'' = \sqrt{\frac{a}{g}} \times \frac{BM'Z}{h}\left(\frac{h}{8\,a} + \frac{9\,h^2}{256\,a^2}\right).$$

But, by taking π for the ratio of the circumference of a circle to its diameter, we have $\dfrac{BM'Z}{h} = \dfrac{\pi}{2}$; accordingly,

$$t'' = \pi\sqrt{\frac{a}{g}}\left(\frac{h}{8\,a} + \frac{9\,h^2}{256\,a^2}\right).$$

Comparing this value of t'' with that of t' found above, we shall have,

$$t' : t'' :: \pi\sqrt{\frac{a}{g}} : \pi\sqrt{\frac{a}{g}}\left(\frac{h}{8\,a} + \frac{9\,h^2}{256\,a^2}\right),$$

or

$$t'' = t'\left(\frac{h}{8\,a} + \frac{9\,h^2}{256\,a^2}\right),$$

a quantity in which $\dfrac{h}{a}$ is the versed sine of the arc described during a semioscillation, radius being 1.

Suppose t' equal to $1''$, and that the arcs described on each side of the vertical are 5°. The versed sine of 5° is 0,0038053, radius being 1. Consequently, $\frac{h}{8a} = 0{,}0004757$. With respect to the term $\frac{9h^2}{256\,a^2}$, it is less than a unit of the sixth place. The error in each oscillation will, therefore, be

$$t'' = 1'' \times 0{,}0004757 = 0'',0004757.$$

Thus, if a body descend by the action of gravity along a circular curve, and describe arcs infinitely small on each side of the lowest point in a second of time, the duration of each oscillation, no allowance being made for friction or the resistance of the air, would differ only $0'',0004757$ from that of an oscillation through an arc of 5° on each side of the lowest point, so that in a day or during $24 \times 60 \times 60 = 86400''$ vibrations, the difference would amount to $86400 \times 0'',0004757$ or $41''$. Thus a pendulum of the length required to vibrate seconds, and performing its oscillations through arcs of 5° on each side of a vertical, would lose only $41''$ a day, when compared with one vibrating in arcs infinitely small.

If the arcs described on each side of the vertical were only 1°, the versed sine of which is 0,0001523, the daily loss would be only $1'',64$, that is, $1\frac{2}{3}$ nearly, and for half a degree, the loss would be $0'',41$ or $\frac{2}{5}$ of a second daily.

1843

Of Pendulums.

344. What we have said is particularly applicable to pendulums. Fig. 165.
By a *pendulum*, is to be understood a rod or thread suspended at one extremity from a fixed point, and supporting at the other extremity one or several bodies. It is called a *simple pendulum* when it is supposed to consist merely of a mass or weight sustained by a thread or rod without gravity, and when at the same time this mass is of a diameter very small relative to the length of the pendulum. We shall speak for the present only of the simple pendulum.

When the pendulum CB is drawn from its vertical position, the force of gravity acting according to the vertical line AM is not wholly employed in moving the body; a part is exerted against the point C. Let therefore the whole force of gravity, represented by AM, be decomposed into two others, represented the one by AN, directed according to CAN, which will be destroyed, and the other by AP which urges the body along the arc AB. Now as the radius CA is perpendicular to the arc, it will be seen that the motion is here decomposed in the same manner as in the case above considered, where the body is supposed, without any material connexion with C, to descend along the arc AB, which has for its radius the length AC of the pendulum. Accordingly every thing which we have said is applicable to pendulums. The following are some of the consequences which are derived from the preceding investigation.

345. We have found for the duration t of an oscillation, the following expression, namely,

$$t = \pi \sqrt{\frac{a}{g}}.$$

Hence, for another pendulum whose length is a', and which is urged by a different force of gravity, or one that is capable of giving the velocity g' in a second, we shall have, by calling t' the duration of an oscillation in this second case,

$$t' = \pi \sqrt{\frac{a'}{g'}};$$

hence we derive the proportion,

$$t : t' :: \pi \sqrt{\frac{a}{g}} : \pi \sqrt{\frac{a'}{g'}} :: \sqrt{\frac{a}{g}} : \sqrt{\frac{a'}{g'}} :: \frac{\sqrt{a}}{\sqrt{g}} : \frac{\sqrt{a'}}{\sqrt{g'}},$$

that is, *if two pendulums of different lengths are urged by different gravities, the durations of the oscillations are as the square roots of the lengths of the pendulums, divided by the square roots of the quantities which denote these gravities.*

346. As gravity is the same in the same place, we shall have for pendulums of different lengths vibrating in the same place or same part of the earth, $g' = g$, and consequently in this case the proportion becomes

$$t : t' :: \sqrt{a} : \sqrt{a'};$$

that is, *in the same place the durations of the oscillations are as the square roots of the lengths of the pendulums.*

347. But if the same pendulum be successively exposed to the action of two different gravities, a being equal to a', the proportion

$$t : t' :: \sqrt{\frac{a}{g}} : \sqrt{\frac{a'}{g'}}$$

becomes

$$t : t' :: \sqrt{\frac{1}{g}} : \sqrt{\frac{1}{g'}} :: \sqrt{g'} : \sqrt{g};$$

in other words, *the durations of the oscillations of the same pendulum in different places are inversely as the square roots of gravity.*

348. Let n be the number of oscillations or *vibrations* made by the pendulum a in a given time, as one hour or $3600''$, we shall have $t = \frac{3600''}{n}$. For the same reason, if we represent by n' the number of vibrations made by the pendulum a' in the same time, we shall have $t' = \frac{3600''}{n'}$;
accordingly,

$$t : t' :: \frac{3600''}{n} : \frac{3600''}{n'} :: n' : n;$$

that is, the numbers of vibrations made in the same time by two pendulums of different lengths are inversely as the durations of their respective vibrations. Consequently, since

$$t : t' :: \sqrt{\frac{a}{g}} : \sqrt{\frac{a'}{g'}},$$

$$n : n' :: \sqrt{\frac{a'}{g'}} : \sqrt{\frac{a}{g}},$$

that is, *the number of vibrations made in the same time by two pendulums of different lengths, and which are urged by different gravities, are in the inverse ratio of the square roots of the lengths of the*

pendulums divided by the square roots of the gravities; so that if the gravities are the same, the number of vibrations will be reciprocally as the square roots of the lengths of the pendulums; and if the lengths are the same, the number of vibrations will be directly as the square roots of the gravities.

349. Hence if the same pendulum, carried to different parts of the earth, does not make the same number of vibrations in the same interval of time, it is to be inferred that gravity is not the same in these places, and the number of vibrations actually made in the same time by the same pendulum in two different places, will furnish the means of ascertaining the relative intensities of gravity at these places. It is by experiments of this kind, taken in connection with the foregoing proposition, that we are now assured of the diminution of gravity as we approach toward the equator, and on the other hand of its augmentation as we proceed from the equator toward either pole.

350. If we call t the time employed by a heavy body, falling freely, in describing the diameter BD or $2\ a$, we shall have

273. $$2\ a = \frac{g\ t^2}{2} \text{ or } \frac{a}{g} = \frac{t^2}{4};$$

Fig. 164. whence

$$\sqrt{\frac{a}{g}} = \tfrac{1}{2}\ t.$$

Substituting this value in the equation,

$$t' = \pi \sqrt{\frac{a}{g}},$$

we obtain

$$t' = \tfrac{1}{2}\ \pi\ t \quad \text{or} \quad \tfrac{1}{2}\ t' = \tfrac{1}{4}\ \pi\ t,$$

which gives

$$\tfrac{1}{2}\ t' : t :: \tfrac{1}{4}\ \pi : 1;$$

that is, the duration of the descent through any small arc AB is to the time of falling through the diameter, as the fourth of the circumference of a circle is to its diameter. But the fourth of the
Geom. circumference is less than the diameter; consequently a body
292.

employs less time in descending along a small arc of a circle of which the inferior tangent is horizontal, than it would employ in falling through the diameter; and since the time required to pass through the diameter is the same with that required to describe any chord AB, it will be seen, that a body would pass sooner from A to B, by descending along the arc AB, than by moving through the straight line AB. Therefore, although the straight line is indeed the shortest way from one point to another, it is not that which requires the least time for the passage of a heavy body. 332.

Of the Line of swiftest Descent. Omit to 368.

351. Not only is not a straight line that along which a heavy body would proceed in the shortest time from one point to another, out of the same vertical, but it is not the arc of a circle which answers to this description; it is the arc of another curve which may be found in the following manner.

Suppose AMR to be the curve sought, or that through which Fig. 166. a heavy body would pass in the least time from a given point A, to a given point B. If we take in this curve two points M, m', infinitely near to each other, the arc $M\,m'$ must also be described in less time than any other arc passing through these same points M, m', since these two points may be taken as the very points in question. Having taken the point N infinitely nearer to $M\,m'$ than M is to m', suppose infinitely small straight lines $M\,N$, $N\,m'$ to be drawn; since the time of describing $M\,m\,m'$ must be a minimum, it follows that the difference between the time of passing through $M\,m\,m'$ and the time through $MN\,m'$, which is the differential of the time, must be zero.

Through the points M, N, m', draw the horizontal lines MP, $m\,P'$, $m'\,P''$, and through A, the vertical line AC. Call AP, x; PM, y; AM, s; and suppose $M\,m = m\,m'$, or that, $d\,s$ is constant. Then $m\,r = d\,x$, $r\,M = d\,y$, $m\,r' = d\,x + d\,d\,x$, $r'\,m' = d\,y + d\,d\,y$. Let u be the velocity with which the body describes $M\,m$; it will be the velocity with which MN is described; and $u + d\,u$ will be that with which $m\,m'$ and $N\,m'$ will be described.

Therefore the time of passing through $M\,m$ will be $\frac{d\,s}{u}$, and the
24. time through $m\,m'$ will be $\frac{d\,s}{u+d\,u}$.

From the points M and m' as centres, and with the radii MN, $m\,m'$, describe the arcs $N\,n$, $m\,t$; then comparing the triangles $N\,m\,n$, $N\,m\,t$, with the triangles $M\,m\,r$, $m\,m'\,r'$, we shall have

$$n\,m = N\,m \times \frac{d\,y}{d\,s},$$

and

$$N\,t = N\,m \times \frac{d\,y + d\,d\,y}{d\,s}.$$

Whence

$$MN = d\,s - N\,m \times \frac{d\,y}{d\,s},$$

and

$$N\,m' = d\,s + N\,m \times \frac{d\,y + d\,d\,y}{d\,s}.$$

Therefore the time through MN will be

$$\frac{d\,s - N\,m \times \frac{d\,y}{d\,s}}{u},$$

and the time through $N\,m'$ will be,

$$\frac{d\,s + N\,m \times \frac{d\,y + d\,d\,y}{d\,s}}{u+d\,u}.$$

We have, therefore,

$$\frac{d\,s - N\,m \times \frac{d\,y}{d\,s}}{u} + \frac{d\,s + N\,m \times \frac{d\,y + d\,d\,y}{d\,s}}{u+d\,u} - \frac{d\,s}{u} - \frac{d\,s}{u+d\,u} = 0,$$

an equation which reduces itself to

$$\frac{N\,m}{d\,s}\left(\frac{d\,y + d\,d\,y}{u+d\,u} - \frac{d\,y}{u}\right) = 0, \text{ or } d\left(\frac{d\,y}{u}\right) = 0\,;$$

then, by integrating, we have

$$\frac{d\,y}{u} = \frac{d\,s}{C}, \text{ or } C\,d\,y = u\,d\,s.$$

But since the velocity u is equal to that which the body would acquire in falling from the height AP, we have

$$u^2 = 2\,g\,x.$$ 277.

Therefore,

$$C\,d\,y = d\,s\,\sqrt{2\,g\,x},$$

and

$$C^2\,d\,y^2 = 2\,g\,x\,d\,s^2 = 2\,g\,x\,(d\,x^2 + d\,y^2)\,;$$

whence we deduce

$$d\,y = \frac{d\,x\,\sqrt{2\,g\,x}}{\sqrt{C^2 - 2\,g\,x}}.$$

To determine the constant C, it will be observed that when $\sqrt{2\,g\,x} = C$, we have $d\,y = d\,s$, answering to the lowest point R, of the curve, where $d\,x = 0$, and $x = h$; therefore if we call v the velocity which the body will have at the point where $\sqrt{2\,g\,x} = C$, the equation $C\,d\,y = u\,d\,s$ becomes $C\,d\,s = v\,d\,s$, which gives $C = v$. And if we call h the corresponding height AC, we shall have

$$v^2 = 2\,g\,h\,;$$ 277

hence

$$C^2 = 2\,g\,h.$$

Therefore

$$d\,y = \frac{d\,x\,\sqrt{2\,g\,x}}{\sqrt{2\,g\,h - 2\,g\,x}} = \frac{d\,x\,\sqrt{x}}{\sqrt{h - x}}$$

is the equation of the curve. But the better to understand this curve, let us give another form to the equation.

Imagine the vertical line RD drawn through the point R, where $d\,y = d\,s$; and having produced PM to O, call AD, a; OR, x'; and OM, y'. Then $x = h - x'$, $y = a - y'$, $d\,x = -\,d\,x'$, $d\,y = -\,d\,y'$; substituting these values, we have

$$d\,y' = \frac{d\,x'\,\sqrt{h - x'}}{\sqrt{x'}} = \frac{h\,d\,x' - x'\,d\,x'}{\sqrt{h\,x' - x'^2}}$$

$$= \frac{\frac{1}{2}\,h\,d\,x' - x'\,d\,x'}{\sqrt{h\,x' - x'^2}} + \frac{\frac{1}{2}\,h\,d\,x'}{\sqrt{h\,x' - x'^2}}.$$

Therefore

$$y' = C' + \sqrt{h\,x' - x'^2} + \int \frac{\frac{1}{2}\,h\,d\,x'}{\sqrt{h\,x' - x'^2}}.$$

Imagine that upon DR or h, as a diameter, is described the semicircle DER. We shall have $OE = \sqrt{h\,x' - x'^2}$, and the arc

Cal. 117.

$$RE = \int \frac{\frac{1}{2}\,h\,d\,x'}{\sqrt{h\,x' - x'^2}};$$

we have therefore generally,

$$OM = C' + OE + RE.$$

To determine the constant C', it must be observed that when $x' = 0$, we have $y' = 0$. Therefore, since OE and RE then become zero, we have $C' = 0$; consequently $OM = OE + RE$; and the curve sought is therefore the common semicycloid, of
Cal. 36. which DER is the generating circle, and AD the semibase.

The only thing which remains to be determined, is the quantity h; for the only things given are the two points A and B, through which the body is to pass. h is determined in this manner.

Having drawn the vertical BK, which meets in K the horizontal line AK passing through the point A, we describe upon AK as a semibase, the semicycloid AVT, that is, a semicycloid of which the generating circle has AK for the length of its semicircumference. And having drawn AB cutting this cycloid in V, we draw VK, and parallel to VK, through the point B, we draw BD, which determines AD for the semibase of the cycloid sought, that is, for the semicircumference of its generating circle. This construction is founded upon the circumstance, that the cycloids AVT, ABR, which have their bases upon AD, and the point A common, are similar, as may be easily shown.*

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* Since the diameters of circles are as their circumferences, or as their semicircumferences;

Geom. 287.

$$DR : KT :: DA : KA;$$

but

$$DR : KT :: DB : KV,$$

hence

$$DB : KV :: DA : KA.$$

352. We have supposed the body to have no velocity on its leaving the point A. But if it had already acquired a certain velocity in a given direction, the origin of the curve would be at some higher point. The equation $C\,d\,y = u\,d\,s$, found above, gives $\frac{u}{C} = \frac{d\,y}{d\,s}$; whence the constant C must be such that, the initial velocity being divided by it, the quotient will be equal to the sine of the angle made by the direction of this initial velocity with the vertical, a condition, which, with the other, that the body must pass through A and B, will determine the cycloid for the case in question.

353. Besides this property of being the curve of swiftest descent in an unresisting medium, the cycloid is on several other accounts quite remarkable. It has, for example, this singular property, that whatever be the point X, from which a body begins to descend along the concave part of the curve, it arrives always at the lowest point R in the same time. This property is thus proved.

Calling t the time, and s the arc RM corresponding to any point M, where the body is found at the end of the time t, we have $d\,t = -\frac{d\,s}{u}$. Now designating by h' the height of X above the horizontal line OM, we have $u = \sqrt{2\,g\,(h' - x')}$. More- 277.
over, it is easy to infer from the value of $d\,y'$, found above, that

$$d\,s = \frac{d\,x'\,\sqrt{h}}{\sqrt{x'}};$$

hence

$$d\,t = -\frac{d\,x'\,\sqrt{h}}{\sqrt{2\,g\,(h'\,x' - x'^2)}} = -\sqrt{\frac{h}{2\,g}} \times \frac{1}{\frac{1}{2}\,h'} \times \frac{\frac{1}{2}\,h'\,d\,x'}{\sqrt{h'\,x' - x'^2}};$$

therefore,

$$t = C - \frac{2}{h'}\sqrt{\frac{h}{2\,g}} \times \int \frac{\frac{1}{2}\,h'\,d\,x'}{\sqrt{h'\,x' - x'^2}};$$

whence, reasoning as above, and calling t' the whole time employed 339.
in falling from X to R, we conclude that

$$t' = \frac{2}{h'} \times \sqrt{\frac{h}{2g}} \times \frac{h'\pi}{2},$$

π being the ratio of the circumference to the diameter. Therefore

$$t' = \pi \sqrt{\frac{h}{2g}};$$

that is, the time t' is independent of the height h' from which the body sets out.

Of the Moment of Inertia.

Fig. 167. 354. Let M, M′, M″, be any masses without gravity, and let them be considered as points situated in the same plane with the point F, and connected together, and with the point F, in such a manner as not to admit of any change in their relative distances, or of any motion except about the point F, or about an axis passing through F, perpendicularly to the plane in which they are situated. Let us suppose that these masses receive at the same time impulses according to the lines w, w', w'', directed in the above plane, and such, that if the masses were free, they would have velocities represented by these lines respectively, it is proposed to determine the motion that would ensue.

We decompose, according to the principle of D'Alembert, the
133. velocities w, w', w'', each into two others, one of which shall be effective, and the others such, that if the masses M, M′, M″, had respectively only this velocity, they would remain in equilibrium.

Now it is manifest that the velocities, which the bodies are supposed to have, since they admit only of a rotation about the point F, must be perpendicular to the radii R, R′, R″. Moreover, in order that these velocities may take place, that is, not mutually disturb each other, it is necessary that they should be proportional to these radii, or to the distances respectively from F. Accordingly, the communicated velocities w, w', w'', being decomposed into the effective velocities v, v', v'', and the velocities u, u', u'', with which the masses would be in equilibrium about the point F,

we have

$$v : v' :: R : R', \text{ and } v : v'' :: R : R'' \quad (1).$$

Letting fall from F the perpendiculars a, a', a'', upon the directions of the velocities u, u', u'', we obtain, 63.

$$M \cdot u \cdot a + M' \cdot u' \cdot a' - M'' \cdot u'' \cdot a'' = 0.$$

Now if we let fall also the perpendiculars c, c', c'', upon the directions w, w', w'', we shall have, by article 62,

$$M \cdot u \cdot a + M \cdot v \cdot R = M \cdot w \cdot c,$$

or

$$M \cdot u \cdot a = M \cdot w \cdot c - M \cdot v \cdot R.$$

In like manner,

$$M' \cdot u' \cdot a' = M' \cdot w' \cdot c' - M' \cdot v' \cdot R'.$$
$$M'' \cdot u'' \cdot a'' = M'' \cdot w'' \cdot C'' + M'' \cdot v'' \cdot R''.$$

If from the sum of the two first of these three equations we subtract the last, we shall have

$$M \cdot u \cdot a + M' \cdot u' \cdot a' - M'' . u'' \cdot a'',$$

or 0, equal to the expression below, thus,

$$\begin{aligned} 0 = \; & M \cdot w \cdot c + M' \cdot w' \cdot c' - M'' \cdot w'' \cdot c'' \\ & - M \cdot v \cdot R - M' \cdot v' \cdot R' - M'' \cdot v'' \cdot R''. \end{aligned}$$

But the above proportions (1) give $v' = \frac{v \cdot R'}{R}$ and $v'' = \frac{v \cdot R''}{R}$;

substituting these values for v' and v'', the equation becomes

$$\begin{aligned} 0 &= M \cdot w \cdot c + M' \cdot w' \cdot c' - M'' \cdot w'' \cdot c'' \\ &\quad - M \cdot v \cdot R - \frac{M' \cdot v \cdot R'^2}{R} - \frac{M'' \cdot v \cdot R''^2}{R} \\ &= M \cdot w \cdot c \cdot + M' \cdot w' \cdot c' - M'' \cdot w'' \cdot c'' \\ &\quad - \frac{(M \cdot R^2 + M' \cdot R'^2 + M'' \cdot R''^2)}{R} v; \end{aligned}$$

whence

$$v = \frac{M \cdot w \cdot c + M' \cdot w' \cdot c' - M'' \cdot w'' \cdot c''}{M \cdot R^2 + M' \cdot R'^2 + M''' \cdot R''^2} \times R.$$

Now the numerator of this fraction, since it expresses the sum of the moments of the forces M · w · c, &c., is equal to the moment of their resultant. If, therefore, we call this resultant ϱ, and its distance from the point F, D; the sum of the moments will be $\varrho \times D$. Moreover the denominator of the above fraction, being the sum of the products of each mass or particle into the square of its distance from F, if we represent in general any one whatever of these masses by m, and its distance from F by r, the sum of these products may be represented by the abridged expression $\int m\, r^2$, ($\int$ denoting the word *sum*); we have accordingly for the velocity of any given point M, whose distance from the axis F is FM or R, the following expression,

$$v = \frac{\varrho \times D}{\int m\, r^2} \times \mathrm{R};$$

also,

$$\varrho \times D = \frac{v}{\mathrm{R}} \int m\, r^2.$$

355. Although we have supposed that all the forces, and all parts of the system are in the same plane, it will be perceived that we should arrive at the same result, if they were in planes parallel to each other, and perpendicular to the axis of rotation, provided that all parts of the system admit only of a rotation about a fixed axis.

356. Accordingly, as a solid body of whatever figure may be considered as an assemblage of material points, thus connected together, we may say generally, that *when a body* L *of whatever fig-*
Fig. 168. *ure, and urged by forces of whatever number and magnitude, can have no other motion, except a motion of rotation about a fixed axis* AB, *situated within or without the body, the velocity belonging to any given point is found by taking the sum of the moments of all the forces (or the moment of the resultant), dividing this sum by the sum of the products of the several parts of the body into the squares of their distances respectively from the axis of rotation, and multiplying the quotient by the distance of the point in question from this same axis.*

Fig. 169. 357. Let G be the centre of gravity of the body L, and let us suppose that while any point M, in turning, describes during an instant, the infinitely small arc v, the centre of gravity G

would describe the arc GG' perpendicular to FG; through the point G' let the line $G'K$ be drawn parallel and equal to GF. Instead of supposing the body to turn about F, we may imagine it carried parallel to itself with a velocity equal to GG', and that at the same time its several parts turn about a movable point G with such a velocity that by taking $G'K = GF$, the point K would describe the arc KF, equal to $G'G$; for, on this supposition, the point F of the body L would still remain stationary. Now the body in this case being free, the resultant of all the motions of rotation about the movable point G is zero. Consequently the result- 79.
ant of all the motions with which the body is actually urged is no other than that which the body L would have, impressed with the velocity GG'; that is, this force must be perpendicular to FG, and equal to

$$L \times GG',$$

the mass of the body being denoted by L. Now since the parts of the body describe similar arcs, we have

$$FM : FG :: v : GG' = \frac{FG \times v}{FM};$$

therefore, the resulting force of all the motions of rotation about the point F, is

$$\frac{L \times FG \times v}{FM}.$$

But although this resultant is the same as if, the body being free, the centre of gravity had received the velocity GG', still it will be seen that this resultant does not pass through G, but through some point O of FG produced; since, the more remote parts having the greater force, the resultant, while it falls on the same side of F with the centre of gravity, must pass at a greater distance from F than this centre. Designating this distance FO at which the resultant passes by D', we shall have for the moment of the resultant

$$\frac{L \times FG \times v}{FM} \times D'.$$

If now, at the instant when the forces $m \cdot w \cdot c$, &c., above con- 354.
sidered, begin to act upon the parts of the body, there be opposed to them, at the distance D', a force equal to that just determined;

that is, equal to the whole effort which the abovementioned forces would exert upon the body, there would evidently be an equilibrium; but in this case the moment $\frac{L \times FG \times v}{FM} \times D'$ must be equal to the moment $\varrho \times D$; accordingly, since

$$\varrho \times D = \frac{v}{MF} \int m\, r^2,$$

we shall have

$$\frac{L \times FG \times v}{FM} \times D' = \frac{v}{FM} \int m\, r^2,$$

and, consequently,

$$D' = \frac{v \times \int m\, r^2 \times FM}{FM \times L \times FG \times v} = \frac{\int m\, r^2}{L \times FG}.$$

358. We hence derive the general conclusion, that, *if any number whatever of forces, directed in any manner we please, in planes perpendicular to the axis of rotation, act upon a body, and are capable of producing only a motion about this axis;* (1.) *The force thus exerted, will be equal to the mass of the body multiplied by the velocity belonging to the centre of gravity;* which velocity is determined by article 357. (2). *This force will be perpendicular to the plane passing through the axis and the centre of gravity.* (3.) *Its distance from the axis (always the same, whatever be the forces and their directions) will be equal to the sum of the products of the several particles of the body into the squares of their distances respectively from the axis, divided by the product of the mass of the body into the distance of the centre of gravity from this same axis.*

359. v denoting always the velocity with which a determinate point M of the body L tends to turn in virtue of the action of any number of forces, or of their resultant ϱ, if we designate the distance of any particle from the axis of rotation by r, and the mass of this particle by m, since $FM : v :: r : \frac{r\, v}{FM}$, we shall have $\frac{r\, v}{FM}$ for the velocity of rotation of the particle m, and $\frac{m\, r\, v}{FM}$ for the force it would exert, and consequently for the resistance it would oppose to ϱ by its inertia; accordingly,

$$\frac{m\ v\ r}{FM} \times r, \text{ or } \frac{m\ r^2\ v}{FM}$$ 60.

will be the moment of this resistance; therefore the sum of the moments of these resistances which the particles of L would oppose to the motion of rotation, produced by ϱ, upon these particles, is $\frac{\int m\ r^2\ v}{FM}$, or $\frac{v}{FM} \int m\ r^2$, for the two expressions are the same, since v and FM do not change, whatever be the particle m, which we consider.

We hence perceive, that, other things being the same, the resistance which the particles of a body oppose to the motion of rotation, communicated to them, is so much the greater as $\int m\ r^2$ is greater.

The quantity $\frac{v}{FM} \int m\ r^2$ is called *the moment of inertia* of a body, and $\int m\ r^2$ *the exponent of the moment of inertia.*

360. We shall see soon how the exponent of the moment of inertia in any body may be determined; but when this exponent has been determined with respect to any axis whatever, it is very easy thence to infer, what it must be with respect to any other axis parallel to the former.

Let AB be any axis, and $A'B'$ another axis parallel to it, and Fig. 170. passing through the centre of gravity G of the body. Let m be any particle of this body; and through m suppose a plane $m\ FF'$, perpendicular to the two axes AB, $A'B'$; $m\ F$, $m\ F'$, being drawn, and the perpendicular $m\ P$ being let fall upon FF', the lines $m\ F$, $m\ F'$, will be perpendicular respectively to AB, $A'B'$. Geom. 313.

This being supposed, we shall have,

$$\overline{m\ F}^2 = \overline{m\ F'}^2 + \overline{FF'}^2 + 2\ FF' \times F'P;$$

hence, Geom. 192.

$$\int m \cdot \overline{m\ F}^2 = \int m \cdot \overline{m\ F'}^2 + \int m \cdot \overline{FF'}^2 + \int m \cdot 2\ FF' \times F'P.$$

Now, since the distance FF' is always the same, whatever be the particle m under consideration, $\int m \cdot \overline{FF'}^2$ is simply $\overline{FF'}^2 \cdot \int m$, or

$\overline{FF'}^2 \times L$, the mass of the body being represented by L. For the same reason $\int m \times 2\,FF' \times F'P$ is simply $2\,FF' \int m \times F'P$. But $\int m \times F'P$, being the sum of the products of the particles into their respective distances from a plane passing through $A'B'$,
64. that is, through the centre of gravity, must be equal to zero; we have therefore simply,

$$\int m \cdot m\overline{F}^2 = \int m \cdot m\overline{F'}^2 + L \times \overline{FF'}^2.$$

Hence, *knowing the exponent* $\int \mathrm{m} \cdot \mathrm{m}\overline{\mathrm{F}'}^2$ *of the moment of inertia with respect to an axis passing through the centre of gravity, we have the exponent with respect to any other axis parallel to this, by adding to the first the product of the mass into the square of the distance between the two axes.*

354. From this result, and the expression for the velocity of rotation, it may be inferred that *of all the axes about which a body may be made to turn in virtue of any force or impulse, those about which the velocity of rotation will be the greatest are such as pass through the centre of gravity;* since the exponent of the moment of inertia with respect to an axis passing through the centre of gravity, is less than it is with respect to any other axis.

Of the Centre of Percussion and the Centre of Oscillation.

361. The foregoing propositions will be found to be of the greatest importance in many inquiries to be resumed hereafter; we shall confine ourselves for the present to the use that may be made of them in finding the *centre of percussion*, and *centre of*
Fig. 171. *oscillation*, of bodies that admit only of a rotation about a determinate axis or point. We understand by the centre of percussion, the point O of the straight line FG, where it would be necessary to place a body in order that it might receive the greatest impression from the body L turning about F. Now it is evident that this point must be that through which passes the resultant of the motions of rotation of all the particles in L. The centre of percussion, therefore, is determined by the proposition of article 358.

As to the centre of oscillation, it is the point O of a body L or Fig. 171.
system of bodies, whose distance from F is equal to the length which a simple pendulum must have in order to perform its oscillations in the same time. We shall see that this centre is the same as the centre of percussion.

Indeed, when the question relates to gravity, the force ϱ, resulting from the action of gravity, exerted upon each material particle of a body, is equal to the whole mass multiplied by the velocity communicated by gravity in an instant to each particle; that is,

$$\varrho = u \times L,$$

u representing this velocity. Moreover, this resultant ϱ passes through the centre of gravity; and consequently its perpendicular distance from the fixed point F, or from the axis passing through F, is FH; hence the velocity of rotation v, of any point M, when the body is left to the action of its gravity, is 354.

$$v = \frac{u \times L \times FH}{\int m\, r^2} \times FM;$$

so that, for the centre of gravity G, the velocity is

$$GG' = \frac{u \times L \times FH}{\int m\, r^2} \times FG.$$

Now in order that a simple pendulum, whose length is FO, may make its oscillations in the same time with the body L, it is necessary, L being supposed to be drawn from a vertical position by the same angular quantity, that the velocity impressed by gravity at O (fig. 172), perpendicularly to FO, should be the same as that of the point O (fig. 171); in other words, that it should be to the velocity of G (fig. 171), as FO is to FG. Now by decomposing the velocity u or OP (fig. 172), communicated by gravity in an instant to a free body, into two others, namely OK in the direction of FO, and OO' perpendicular to FO, we shall have

$$u : OO' :: FO : OZ :: FG : FH;$$

whence

$$u : OO' :: FG : FH,$$

and consequently

$$OO' = \frac{u \times FH}{FG}.$$

We hence derive the proportion,

$$\frac{u \times FH}{FG} : \frac{u \times L \times FH}{\int m\, r^2} \times FG :: FO : FG,$$

which gives

$$FO = u \times FH \times \frac{\int m\, r^2}{u \times L \times FH \times FG} = \frac{\int m\, r^2}{L \times FG},$$

357. which is the same as the expression for the centre of percussion.

362. Since all the forces which act upon the body L, or upon a system of bodies that admit only of a motion of rotation about a point or a fixed axis, cause in this body such a velocity that, for any given point M, we have

$$v = \frac{\varrho \times D}{\int m\, r^2} \times FM,$$

and since it is evident, that if this body were to turn in the opposite direction with the same velocity, there would be an equilibrium among all these forces; we infer, that if a body, turning with a velocity which for a determinate point M is equal to v, would have its motion counterbalanced by a power ϱ, the direction of which passes at a distance from F equal to D, this power taken in connexion with its distance D, must be such that the moment $\varrho \times D$ shall be equal to the velocity of the point M, divided by the distance FM, and multiplied by the sum of the products of the particles into the squares of their distances respectively from F, or from the axis passing through F. Indeed this power must be such as will be sufficient to produce the same velocity in the body L, supposed at rest; and this velocity would be

$$v = \frac{\varrho \times D}{\int m\, r^2} \times FM,$$

which gives

$$\varrho \times D = \frac{v}{FM} \int m\, r^2.$$

363. If a body L, of any figure whatever, admitting only of a motion about a fixed point F, or about an axis passing through F, which may in other respects be situated as we choose, if, I say, a body L be struck by a body N, the motion of each after collision may be determined by the principles above established. Fig. 173.

Thus, let u be the velocity of N, before collision, according to the perpendicular TH, and u' its velocity after collision; $u - u'$ will be the velocity, and $N(u - u')$ the quantity of motion, lost by collision, and which will pass into the body L. This quantity of motion will cause in L a velocity of rotation, such that the point T, for example, will turn with a velocity 290.

$$v = \frac{N(u - u') \times FH}{\int m r^2} \times FT \qquad (1), \qquad 354.$$

FH being drawn perpendicular to TH. Let the infinitely small arc TT', described about the centre F, represent this velocity; the parallelogram $TA\,T'C$ being formed upon the tangent TA and the perpendicular TH, it will be seen, by substituting for TT' the velocities TA, TC, that the velocity TA cannot affect the velocity u' which the body N must have; but that the velocity TC would impair the velocity u' if it were smaller than u'; accordingly, since we suppose that u' is actually the velocity which N preserves after collision, it is necessary that TC should be equal to u'. Now the similar triangles FHT, TCT', give,

$$FT : FH : TT' \text{ or } v : TC,$$

whence,

$$\frac{v \times FH}{FT} = TC = u',$$

and consequently

$$v = \frac{u' \times FT}{FH}.$$

Substituting for v this value in the equation (1), we have

$$\frac{u' \times FT}{FH} = \frac{N(u - u') \times FH}{\int m r^2} \times FT,$$

from which we deduce the value of u'; thus,

$$\frac{u'}{FH} = \frac{N \times u \times FH}{\int m\, r^2} - \frac{N \times u' \times FH}{\int m\, r^2},$$

$$\frac{u' \times \int m\, r^2}{\int m\, r^2 \times FH} + \frac{N \times u' \times \overline{FH}^2}{\int m\, r^2 \times FH} = \frac{N \times u \times FH}{\int m\, r^2},$$

$$u' \left(\frac{\int m\, r^2 + N \times \overline{FH}^2}{\int m\, r^2 \times FH}\right) = \frac{N \times u \times FH}{\int m\, r^2},$$

$$u' = \frac{N \times u \times FH}{\int m\, r^2} \times \frac{\int m\, r^2 \times FH}{\int m\, r^2 + N \times \overline{FH}^2},$$

$$= \frac{N \times u \times \overline{FH}^2}{\int m\, r^2 + N \times \overline{FH}^2}.$$

From this the value of v, or the velocity of rotation, is readily obtained. But the equation $v = \frac{u' \times FT}{FH}$, or $v\, FH = u'\, FT$, since it gives the proportion

$$u' : v :: FH : FT,$$

makes it evident, that u' is the velocity of rotation of the point H; from which it will be seen, that the point H turns with the velocity that remains to N after collision.

364. We hence perceive, that in order to find the motion of bodies that turn about a fixed point or axis, we must be able to determine the value of $\int m\, r^2$. This will always be easy, as we shall soon show, when the bodies are such as admit of being expressed by equations. We may, indeed, in any case consider the body as composed of parallelopipeds, pyramids, &c., which are capable of being thus expressed; and finding for each component part the value of $\int m\, r^2$, take the sum of these as the total value of $\int m\, r^2$ for the entire body or system of bodies.

When the body is such as admits of being expressed by an equation, we proceed thus in finding the value of $\int m\, r^2$.

Let AB be the axis of rotation, and through AB suppose two planes PQ, AR, to pass perpendicularly to each other; let m be any particle of the body in question, and having let fall the perpendicular $m\,F$ upon AB, we draw $m\,H$ perpendicularly to the plane RA; and joining FH, this line will be perpendicular to AB, and consequently to the plane PQ. The right angled triangle $m\,HF$ gives Fig. 174. Geom. 186.

$$\overline{Fm}^2 = \overline{FH}^2 + \overline{Hm}^2;$$

whence,

$$\int m \times \overline{Fm}^2 \text{ or } \int m\,r^2 = \int m \times \overline{FH}^2 + \int m \cdot \overline{Hm}^2.$$

The problem, therefore, reduces itself to finding the sum of the products of the particles into the squares of their distances from two planes, which pass through the axis of rotation, and are perpendicular to each other. Now, when the algebraic expression for this sum is found with respect to one of the planes, it is easily obtained with respect to the other. Let us therefore inquire how we can find the sum of the products of the particles of a body into the squares of their distances respectively from a known plane.

We will suppose the body divided into infinitely thin strata, parallel to the given plane; and, representing the thickness of one of these strata by DD', its surface by σ, and its distance FD Fig. 175. from the plane in question by x, since the points of the surface σ are all distant from the plane PQ by the same quantity x, we shall have $x^2\,\sigma\,d\,x$ for the sum of the products of all the points of this surface into the squares of their distances respectively from this plane, and consequently $\int x^2\,\sigma\,d\,x$ for the entire sum of these products for the whole body.

If we represent, in like manner, by x' the corresponding distances from the plane perpendicular to PQ, and passing through the axis of rotation AB (the body being supposed to be divided into strata parallel to this second plane), and by σ' the surface of one of these strata, we shall have $\int x'^2\,\sigma'\,d\,x'$ for the sum of the products of the particles into the squares of their distances respectively for this second plane; and accordingly

$$\int x^2 \sigma \, d x + \int x'^2 \sigma' \, d x'$$

will be the value of the sum of the products of each particle of the body into the square of its distance from the axis AB.

Fig. 176. 365. Let us now suppose, by way of illustration, that the body in question is a rectangular parallelopiped, turning about the axis AB perpendicular to the axis of the parallelopiped, and to the side IK. By the nature of this body, the surface σ is constant; thus the integral $\int x^2 \sigma \, d x$ is $\frac{x^3 \sigma}{3}$, which, when x is equal to the altitude h of the parallelopiped, becomes $\frac{h^3 \sigma}{3}$.

In like manner, σ' being a constant quantity, $\int x'^2 \sigma' \, d x'$ becomes

$$\frac{x'^3 \sigma'}{3},$$

or, MN being represented by h', which gives $x' = \frac{1}{2} h'$,

$$\tfrac{1}{8} \times \frac{h'^3 \sigma'}{3};$$

and, as the plane which passes through the axis divides the body into two equal parts, the two halves will be

$$\tfrac{1}{4} \times \frac{h'^3 \sigma'}{3} \text{ or } \frac{h'^3 \sigma'}{12};$$

therefore the entire sum of the products will be

$$\frac{h^3 \sigma}{3} + \frac{h'^3 \sigma'}{12}.$$

357. If we would find the centre of percussion or of oscillation, we have only to divide this quantity by the product of the mass of the parallelopiped into the distance of its centre of gravity; that is,
Geom. 405. by $h \, h' f \times \frac{1}{2} h$ or $\frac{1}{2} h^2 h' f$, IM being denoted by f, which gives for the distance of the centre of percussion or of oscillation

$$\frac{2 h^3 \sigma}{3 h^2 h' f} + \frac{h'^3 \sigma'}{6 h^2 h' f} \text{ or } \frac{2 h}{3} + \frac{h'^2}{6 h},$$

since

$$\sigma = h' f, \quad \text{and} \quad \sigma' = h f.$$

If h' is very small with respect to h, so that $\frac{h'^2}{6\,h}$ may be neglected, the expression becomes $\frac{2\,h}{3}$. Hence, *the distance of the centre of percussion or centre of oscillation of a straight line, or of a parallelogram, turning about one of its sides, as an axis, is $\frac{2}{3}$ of the length from the point of suspension or axis.*

Thus, the rod or bar FA, turning about the fixed point F would strike a nail T with the greatest effect when the distance of the nail FP is equal to $\frac{2}{3}\,FA$. Fig. 177.

If the rod FA be considered as turning by the action of gravity only, the force which it would exert upon the nail, would be equal to the mass of the rod multiplied by the velocity acquired by the centre of gravity G, in falling along $G'G$, that is, by the velocity acquired by a heavy body in falling through the height GD. 339.

366. We take the sphere as a second example. In this case the surface which we have called σ, is a circle, having for its radius IM, which I shall call y; and, π being the circumference of a circle whose diameter is 1, we have Fig. 178.

$$\pi\,y^2 = \sigma. \qquad \text{Geom. 291.}$$

Let DI be denoted by z, and the radius of the sphere by R; we have

$$y^2 = 2\,\mathrm{R}\,z - z^2, \qquad \text{Trig. 101.}$$

and consequently,

$$\sigma' = \pi\,(2\,\mathrm{R}\,z - z^2).$$

Calling DF, a,

$$FI \text{ or } x = z + a, \quad \text{and} \quad d\,x = d\,z\,;$$

consequently,

$$\int x^2\,\sigma\,d\,x$$

becomes

$$\int (z + a)^2 \times \pi\,(2\,\mathrm{R}\,z - z^2)\,d\,z,$$

or, by developing the whole,

$$\int \pi (2a^2 \mathrm{R} z\, dz + 4a\mathrm{R} z^2 dz - a^2 z^2 dz + 2\mathrm{R} z^3 dz - 2az^3 dz - z^4\, dz),$$

Cal. 85. and by integrating, we have

$$\pi \left(a^2 \mathrm{R} z^2 + \tfrac{4}{3} a \mathrm{R} z^3 - \tfrac{1}{3} a^2 z^3 + \tfrac{1}{2} \mathrm{R} z^4 - \tfrac{1}{2} a z^4 - \tfrac{1}{5} z^5\right),$$

which, when $z = 2\,\mathrm{R}$, becomes

$$\pi \left(4\, a^2 \mathrm{R}^3 + \tfrac{32}{3}\, a \mathrm{R}^4 - \tfrac{8}{3}\, a^2 \mathrm{R}^3 + 8\, \mathrm{R}^5 - 8\, a \mathrm{R}^4 - \tfrac{32}{5}\, \mathrm{R}^5\right)$$

or

$$\pi \left(\tfrac{4}{3}\, a^2 \mathrm{R}^3 + \tfrac{8}{3}\, a \mathrm{R}^4 + \tfrac{8}{5}\, \mathrm{R}^5\right).$$

To find the value of $\int x'^2\, \sigma'\, d\, x'$, it is not necessary to begin the calculation again, since from the regular figure of the sphere, it is evident that this value will be similar to the former; we have only to suppose, therefore, that a, which expresses the distance of the plane PQ from the surface, becomes $-\,\mathrm{R}$; that is, that this plane passes through the centre, it being supposed at the same time to be perpendicular to its first position, and we shall have

$$\pi \left(\tfrac{4}{3}\, \mathrm{R}^5 - \tfrac{8}{3}\, \mathrm{R}^5 + \tfrac{8}{5}\, \mathrm{R}^5\right) \text{ or } \pi \times \tfrac{4}{15}\, \mathrm{R}^5.$$

The two integrals being added together, make

$$\pi \left(\tfrac{4}{3}\, a^2 \mathrm{R}^3 + \tfrac{8}{3}\, a \mathrm{R}^4 + \tfrac{28}{15}\, \mathrm{R}^5\right).$$

Geom. 548. Since the bulk of the sphere is $\pi \times \frac{4}{3}\, \mathrm{R}^3$ or $\frac{4}{3}\, \pi\, \mathrm{R}^3$, and the distance of its centre of gravity from the plane PQ is $a + \mathrm{R}$, if we divide the above result by the product $\frac{4}{3}\, \pi\, \mathrm{R}^3 \times (a + \mathrm{R})$ of these two quantities, we shall have the distance of the centre of oscillation
357. and that of percussion; thus,

$$FO = \frac{\pi \left(\frac{4}{3}\, a^2 \mathrm{R}^3 + \frac{8}{3}\, a \mathrm{R}^4 + \frac{28}{15}\, \mathrm{R}^5\right)}{\frac{4}{3}\, \pi\, \mathrm{R}^3\, (a + \mathrm{R})}$$

$$= \frac{a^2 + 2\,a\,\text{R} + \frac{7}{5}\,\text{R}^2}{a + \text{R}}$$

$$= \frac{a^2 + 2\,a\,\text{R} + \text{R}^2 + \frac{2}{5}\,\text{R}^2}{a + \text{R}}$$

$$= a + \text{R} + \tfrac{2}{5} \times \frac{\text{R}^2}{a + \text{R}} = FG + \tfrac{2}{5} \times \frac{\overline{DG}^2}{FG}.$$

Hence the centre of oscillation and that of percussion are below the centre of the sphere; and the centre of the sphere cannot be taken for the centre of oscillation or that of percussion, except when its radius is very small compared with the distance of the centre G from the point of suspension.

If the sphere is suspended by a rod or lamina, and we would have regard to its mass, it will be recollected, that we have 365.
found $\frac{h^3\,\sigma}{3} + \frac{h'^3\,\sigma'}{12}$ for the sum of the products of the particles of such a body into the squares of their distances respectively from the fixed point or axis. Now h is what we have represented by a; moreover, since

$$\sigma = h'\,f, \text{ and } \sigma' = h\,f = a\,f,$$

we shall have by substitution,

$$\frac{a^3\,h'\,f}{3} + \frac{h'^3\,a\,f}{12};$$

this quantity and that for the sphere must be multiplied respectively by the specific gravities S, S', of these bodies, if their specific gravities be different; then by adding the two products, we shall have,

$$S\,\frac{a^3\,h'\,f}{3} + S\,\frac{h'^3\,a\,f}{12} + S'\,\pi\,(\tfrac{4}{3}\,a^2\,\text{R}^3 + \tfrac{8}{3}\,a\,\text{R}^4 + \tfrac{28}{15}\,\text{R}^5)$$

for the sum of the products of the particles of the whole system into the squares of their distances respectively from the axis. This sum divided by the product of the masses, $S\,a\,h'\,f + S'\,\frac{4}{3}\,\pi\,\text{R}^3$ into the distance of the centre of gravity from the axis, gives the distance of the centre of oscillation.

367. It may suffice in practice to divide the body into a great number of parts, and multiply each part by the square of its distance from the axis in order to obtain with sufficient exactness the value of $\int m\, r^2$.

Of the actual Length of the Seconds Pendulum.

368. The number of vibrations performed in the same time by two different pendulums, urged by the same gravity, being inversely as the square roots of the lengths of these pendulums, we can find very nearly the length of the seconds pendulum for any given place by a very simple process. Having suspended to a very fine wire of at least three feet in length, a small dense body, as a ball of lead, gold, or platina, we ascertain the length of this wire and the radius of the ball with great exactness. We then cause this pendulum to vibrate by drawing it a little from a vertical position, and count the number of vibrations performed in a given time, as one hour, very carefully determined, and then make use of the proportion; as 3600, the number of vibrations to be performed by the pendulum sought, is to the number actually performed by the above pendulum, so is the square root of the
346. length of this latter pendulum to a fourth term or x, which will be the square root of the length of the pendulum sought; and by squaring this fourth term, we shall have very nearly the length of the pendulum required to vibrate seconds.

This result would be exact only on the supposition that the wire or string is without weight, and that the ball consists only of a single particle or has its matter concentrated at the centre.

369. If we attempt to find geometrically the centre of oscillation of the ball and wire, we shall still be liable to some small error arising from irregularities in the form and distribution of the matter in question. We accordingly have recourse to another method, depending on a curious property of the compound pendulum by which the distance between the point of suspension and centre of oscillation, answering to the length of the simple pendulum vibrating in the same time, can be ascertained with the greatest precision.

We have obtained a general expression for the distance in question, as follows, namely, Fig.170.

$$FO = \frac{\int m\, r^2}{L \times FG} = \frac{\int m \cdot \overline{m.F}^2}{L \times FF'}. \qquad 361.$$

But

$$\int m \cdot \overline{m\,F}^2 = \int m \cdot \overline{m\,F'}^2 + L \times \overline{FF'}^2; \qquad 360.$$

whence, by substitution,

$$FO = \frac{\int m \cdot \overline{m\,F'}^2 + L \times \overline{FF'}^2}{L \times FF'},$$

that is,

$$FO \text{ or } FF' + F'O = \frac{\int m \cdot \overline{m\,F'}^2}{L \times FF'} + FF',$$

whence

$$F'O = \frac{\int m \cdot \overline{m\,F'}^2}{L \times FF'}.$$

Thus, *the distance of the centre of oscillation below the centre of gravity is equal to the sum of all the parts multiplied by the squares of their respective distances from the axis drawn through the centre of gravity, divided by the product of the mass into the distance of the centre of gravity from the axis of suspension.*

Now by multiplying both members of the above equation by FF', and dividing both by $F'O$, we shall obtain,

$$F'F = \frac{\int m \cdot \overline{m\,F'}^2}{L \times OF'}.$$

Accordingly, if we consider the body as inverted, and make O the point of suspension, F will become the centre of oscillation, since we have the same expression as before for the distance of this point below the centre of gravity.

We hence infer, that *the point of suspension and centre of oscillation are convertible, that is, either being made the point of suspension the other becomes the centre of oscillation.*

Reciprocally, if two points are so chosen, or so adjusted to each other by movable weights, that the pendulous body shall vibrate in the same time when suspended from one as when suspended from the other, these points are alternately the centres of oscillation and points of suspension, and the distance asunder is the length of the pendulum in question, and equal to that of a simple pendulum vibrating in the same time. The above proposition was demonstrated by Huygens, the original author of the theory of the pendulum, but it was not till very lately applied to any useful purpose. Captain Kater was the first to perceive that it furnished a very simple and accurate method of determining the length of the compound pendulum.

Figure 179 represents Captain Kater's pendulum. The axes *F*, *O*, were adjusted by means of intermediate movable weights *C*, *D*, and with so much accuracy that the number of oscillations made in twenty-four hours, *F* being uppermost, differed from those performed in the same time with *O* uppermost, less than half a vibration; and the means of twelve sets of observations with first one then the other uppermost, differed from each other less than the hundredth of a vibration. The length of the pendulum, as thus obtained, is stated to be 39,1386 inches. This is for the latitude of London, or 51° 31′ 08″,04 *N*., and on the supposition of the arcs of vibration being infinitely small, taking place in a vacuum, and at the level of the sea, the temperature being 62° by Fahrenheit's thermometer. This determination exceeds what was considered the most accurate result of the methods previously in use by 0,00813 or nearly one hundredth of an inch, a very important difference in researches where the ten-thousandth of an inch is an appreciable quantity.

It may be observed, moreover, that if the two axes of the pendulum be cylindric surfaces, the points of suspension and oscillation are truly in these surfaces, and the length sought is rigorously the distance between these surfaces. This second property, so necessary to the completeness of the method, when actually applied to practice, was discovered by Laplace. See Ed. Rev. vol. xxx. p. 407. Phil. Trans. for 1818.

370. It is not necessary to go through the same process in order to find the length of a pendulum required to vibrate in any other proposed time, as half a second, or half a minute. The principles we have investigated will enable us to solve all problems of this kind with the greatest facility and exactness, when the length and time of vibration of one pendulum is known. Thus if it is proposed to find the length of a pendulum required to vibrate half 346.
minutes, the proportion

$$t^2 : t'^2 : : a : a',$$

by substituting for t, t', $1''$ and $30''$, and for a 39,1386, the length of the seconds pendulum, we have

$1^2 : : (30)^2 : : 39{,}1386 : a' = 39{,}1386 \times 900 = 35224{,}74$ inches, or 2935,39 feet.

In like manner, the length and time of vibration of one pendulum being known, the time of vibration, in the same place, of any other pendulum whose length is given, may be determined. Suppose, for example, that it is required to find the time in which a pendulum of 20 feet, or 240 inches in length, would perform its vibrations; by substituting the known quantities in the general proportion,

$$\sqrt{a} : \sqrt{a'} : : t : t',$$

we have

$$\sqrt{39{,}1386} : \sqrt{240} : : 1'' : t' = \sqrt{\frac{240}{39{,}1386}} = 2'',5 \text{ nearly}.$$

Measure of the Force of Gravity.

371. It will be easy now to determine through what space a heavy body must pass in the first second of its fall, the air and all other obstacles being removed. For the equation 342.

$$t' = \pi \sqrt{\frac{a}{g}}$$

gives, by squaring both members and transposing,

$$g = \frac{\pi^2 a}{t'^2};$$

in which g represents the velocity acquired by a heavy body at the end of the first second of its fall, and which is double the height or space through which it would descend in a second from
266. a state of rest; a is the length of the pendulum each of whose vibrations is performed in the time t'. Accordingly, if for t' we put one second, a must be 39,1386 inches for the latitude of London.* Moreover π, the ratio of the circumference of a circle to
Geom. 294. its diameter, is equal to 3,1416 nearly; hence

$$g = (3{,}1416)^2 \times 39{,}1386.$$

Accordingly,

3,1416....2 log....0,99430	
39,1386.......log....1,59260	
386,28	2,58690

The value of g, therefore, is 386,28 inches, or 32,19† feet, equal to 32,2 nearly; and half this quantity or 16,1 is the space described by a heavy body in an unresisting medium at the surface of the earth in one second from the commencement of its motion.
273. We have thus fulfilled our promise.

Application of the Pendulum to Time-Keepers.

372. The pendulum attached to clocks for the purpose of reg-
Fig.180. ulating their motions, is ordinarily a rod of metal or wood loaded

* The length of the seconds pendulum, and consequently the value of g, is referred to the latitude of London on account of the great accuracy of the observations that have been made at this place. The difference, however, in the length of the pendulum in different latitudes, at the level of the sea, is so small as to amount only to about $\frac{1}{5}$ of an inch at the extreme, or when the places to which the observations relate are the equator and the pole; and the difference in the value of g at these places, is only about two inches, as may be easily shown by the above formula.

† The most accurate observations on the length of the seconds pendulum at Paris in latitude 48° 51′ give for the value of g 32,182 ft.

at the lower extremity with a weight in the form of a lens, so placed as to meet with as little resistance as possible from the air. The axis also or point of suspension is fitted to have very little friction. Connected with the pendulum, is a train of wheels and pinions, the teeth and leaves of which are so adapted to each other, that the motions correspond to the several divisions of time, and their axes carry indexes that show by the arcs they describe, the hours, minutes, and seconds. Around the axis at one extremity of this train of wheels, is wound a cord bearing a weight, that would put the whole system in rapid motion, but for the appendage to the pendulum CFD at the other extremity of the train of wheels, which, while the pendulum is at rest, effectually prevents all motion. But if the pendulum be made to vibrate, it will suffer one tooth to pass or escape at each vibration, while at the same time the impulse of the teeth upon the arms FC, FD, is so adjusted, by increasing or diminishing the weight, as just to overcome the friction and the resistance of the air, and thus to keep up the motion, while the action of the weight continues. The contrivance by which the train of wheels is connected with the pendulum, is called the *escapement*.

373. On account of the constancy of gravity the oscillations of the pendulum, other things being the same, must be equal or of the same duration. There are, however, several causes that tend to disturb this isochronism. (1.) The air is subject to changes of density, on account of which the arcs of vibration will sometimes be longer and sometimes shorter, while the maintaining power remains the same.* But if these changes are noted, or if the arcs of vibration are noted, the deviation from perfect regularity can be calculated, and allowance made accordingly.† It

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* As the air becomes more dense the pendulum is more resisted and would seem to be retarded, but the arc of vibration being diminished, the clock goes faster, so that one of these causes tends 343.
to counteract the other. In like manner, when the motion of the axis and wheels is obstructed by dust or want of oil, the impulse of the weight communicated to the pendulum is diminished, the arcs of vibration are reduced, and hence there is a tendency to increase its rate of going.

† If the pendulum could be made to move in the arc of a cycloid, it will be perceived from what has been said of this curve, 353.

may be remarked, moreover, that the irregularity from this cause, is rendered for common purposes altogether inconsiderable, by making the pendulum very heavy, and the arcs of vibration very small.

374. (2.) A much more important source of error, in the rate of going of common clocks, is to be referred to changes in the actual length of the pendulum arising from heat and cold. A brass pendulum rod, for instance, has its length increased about two hundredths of an inch for a change of temperature of 30° of Fahrenheit's thermometer. This would seem to be a small quantity; yet as it is continually exerting an influence, the accumulated effect in the course of 24 hours or 86400″ amounts to more than a third of a minute. The expansion of iron is about $\frac{3}{5}$ of that of brass. There are some kinds of wood that are subject to very little variation of length, particularly in the direction of the fibres, on account of temperature. Still no substance is entirely free from these changes. The effect of any augmentation or diminution of length in the pendulum may be computed
346. by means of the principles that have been investigated.

375. But we can obtain more convenient and sufficiently exact formulas for the variation in the rate of the going of a clock, when the changes in the length of the pendulum are very small, as those are which arise from heat and cold. a being the exact length of the seconds pendulum, or that by which the clock would keep correct time, and a' the actual length, as affected by heat and cold, if we put n for the number of oscillations in a day, performed by the former, n' for the corresponding number of the latter, and t' for the time of this latter, we shall have

346.

$$t' = \sqrt{\frac{a'}{a}};$$

also

348.

$$t' = \frac{n}{n'}$$

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that all arcs whether longer or shorter, would be described in the same time. But the practical difficulties attending all the methods hitherto proposed, are such as to occasion errors, that more than compensate for the theoretical advantages to be derived from a cycloidal motion.

whence

$$\frac{n}{n'} = \sqrt{\frac{a'}{a}},$$

and

$$a' = \frac{a\,n^2}{n'^2}.$$

Suppose x to be the augmentation or diminution of length in question, and y the corresponding daily loss or gain in seconds, we shall have

$$a' = a \pm x = \frac{a\,n^2}{n'^2} = \frac{a\,n^2}{(n \mp y)^2} = \frac{a\,n^2}{n^2 \mp 2\,n\,y + y^2} = \frac{a}{1 \mp \frac{2\,y}{n}},$$

nearly, neglecting $\frac{y^2}{n^2}$ as very small; that is,

$$a \pm x = a\left(1 \pm \frac{2y}{n}\right),$$

nearly, from which we obtain,

$$x = \frac{2\,y\,a}{n}, \text{ and } y = \frac{n\,x}{2\,a}.$$

Thus, if for any given rise of the thermometer, the pendulum is lengthened one hundredth of an inch, we shall have for the number of seconds lost per day,

$$y = \frac{n\,x}{2\,a} = \frac{86400'' \times 0{,}01}{2 \times 39{,}14} = 11'' \text{ nearly.}$$

On the other hand, if a clock is known to keep time correctly at a particular temperature, as 55° for instance, and at 32° is found to gain 7″ a day, we should be able to determine the corresponding diminution in length, or the contraction in the rod of the pendulum, answering to this number of degrees; thus,

$$x = \frac{2\,y\,a}{n} = \frac{2 \times 7 \times 39{,}14}{86400} = 0{,}006 \text{ inches.}$$

376. It will be seen, therefore, that by rendering the weight of the pendulum movable upon the rod, and connecting it with a micrometer screw, a correction may be appli d for the expansion

and contraction according to the state of the thermometer. But a more convenient method has been devised by which the expansion of one metal is made to counteract that of another. The expansion of iron and brass being to each other as three to five, Fig.181. if we make the rod FB of iron, and the rod AO of brass in the proportion of 5 to 3; they being connected at the lower extremities, and the weight being attached at O, the rod AO will expand upward just as much as the rod FA expands downward, and the point O where the weight is applied, will consequently remain amid all changes of temperature at the same distance from F, the point of suspension. A number of rods of each kind is usually employed as represented in figure 182, where the rod which supports the weight, is attached at F and free at D, D', the brass rods expanding upward and the iron ones downward as before; so that if the proper proportion as to length be observed, a compensation for the effect of temperature will be obtained. Other means have been invented for accomplishing the same purpose. Of these we shall mention only one which has been attended with great success. The weight AB is made to consist of a glass tube about two inches in diameter, and from 4 to 7 inches long, Fig.183. filled with mercury.* As the rod of the pendulum supporting this weight, expands downward, the mercury expands upward, as in the contrivance first mentioned, and the quantity may be increased or diminished till a compensation is effected. A clock provided with a pendulum of this construction, made by T. Hardy of London, for the Royal Observatory at Greenwich, was found after

* The expansions of glass and mercury being as 1 to 10 very nearly, if the suspending rod be of glass, the column of mercury must be $\frac{1}{10}$ of the length of the pendulum or about 4 inches. If the rod be of iron, as this substance has a greater expansion in the ratio of 3 to 2 nearly, the column of mercury should be about 6 inches. A steel rod would require a column 6,4 inches in length, which, on the supposition of a diameter of two inches, would weigh 10lbs. From accurate calculation, it is found that if such a pendulum should keep perfectly true time, when the thermometer is at 30°, and that it should gain or lose 1″ a day when the thermometer is at 90°, the imperfection would be remedied by the subtraction or addition, as the case required, of 10 ounces of mercury.

two years' trial to vary only $\frac{1}{5}$ of a second in 24 hours from its mean rate of going. A clock of the same construction owned by W. C. Bond of Boston, though much less costly, has been found by careful observation, to go with nearly the same accuracy.

377. A *watch* or *chronometer* differs from a clock by having a spring for its maintaining power, and a horizontal instead of a vertical pendulum, in which a small, fine spring performs the office of gravity. The pendulum or *balance*, in this case also, Fig. 184.
is subject to irregularity from heat and cold, and requires a distinct compensation. Considerable weights m, m', are attached to the balance by means of slips $C\,m$, $C'\,m'$, of brass and steel, the brass slip in each being outermost. While, therefore, the general expansion of the wheel tends to throw the weight to a greater distance, the superior expansion of the brass slip over the steel brings the weight nearer to the centre, and the length of the slip being properly adjusted to the weight, the centre of oscillation, or rather of *gyration*, will be preserved always at the same distance from the axis.

378. We have found formulas for the difference in the rate of going of a clock answering to small changes in the length of the pendulum, the position with respect to the centre of the earth, and consequently the force of gravity, being supposed to remain the same. It will be easy also to find formulas for the variation in the force of gravity and in the rate of the going of a clock, depending upon small changes of distance from the centre of the earth. n, n', for example, being the number of vibrations of the same pendulum at the two stations respectively, the pendulum being supposed to vibrate seconds at the first; from the proportion,

$$n : n' :: \sqrt{\frac{a'}{g'}} : \sqrt{\frac{a}{g}}, \qquad 348.$$

when $a' = a$, we have

$$g' = \frac{g\, n'^2}{n^2}.$$

If the second station be below the first, or that at which the pendulum vibrates seconds, g' will exceed g, and the clock will

gain; on the contrary supposition it will lose. Let

$$g' = g\,(1 \pm x),$$

and let y denote the daily gain or loss in seconds; we shall have

$$g\,(1 \pm x) = \frac{g\,(n \pm y)^2}{n^2} = \frac{g\,(n^2 \pm 2\,y\,n)}{n^2}$$

nearly. Whence,

$$x = \pm\,\frac{2\,y}{n}.$$

Thus, if a pendulum fitted to vibrate seconds at the equator, would, upon being carried to the pole, gain $5'$ or $300''$ a day, we should have

$$x = \frac{2 \times 300}{86400} = \frac{1}{144};$$

that is, the force of gravity at the equator is to that at the pole, on this supposition, as 144 to 145.

Let the difference h in the distances of the two stations from the centre of the earth be given, gravity being supposed to vary inversely as the square of the distance, the gain or loss of the clock might be readily found as follows.

If we call R the distance of the centre of the earth from the first station, and g the force of gravity at this station, the pendulum being supposed to vibrate seconds, we shall have for the distance of the second station $R \pm h$, and for the force of gravity at this station,

$$g\,\frac{R^2}{(R \pm h)^2} = g\left(1 \mp \frac{2\,h}{R}\right)$$

nearly. Hence, putting $\frac{2\,h}{R}$ for x in the above formula, we obtain

$$\mp\,\frac{2\,h}{R} = \pm\,\frac{2\,y}{n}, \text{ and } \pm\,y = \mp\,\frac{n\,h}{R}.$$

Thus, if the second station be above the first, as 1 mile for instance, the radius of the earth being 3956, or 4000 nearly, the

the formula gives

$$-y = \tfrac{86400}{4000} = 21{,}6'',$$

the sign — indicating that the clock loses.

Omit to Art. 400

Rotation of Bodies unconfined.

379. It has been demonstrated that when a body L receives an impulse in a direction HZ, not passing through its centre of gravity G, this impulse is transmitted entirely to the centre of gravity, which moves in a direction parallel to HZ according to which the body has received the impulse; and that the parts of this body, in the mean time, turn about the centre of gravity in the same manner as if it were fixed. Therefore, if the figure of this body, and the forces impressed upon it (of which I suppose ϱ to represent the resultant) are such that it can turn only about a single axis; as this axis will necessarily pass through the centre of gravity, all that we have said on the subject of the moment of inertia, is applicable to this case, understanding by r in $\int m\,r^2$, the distance of any particle from the axis which passes through the centre of gravity, and by $\varrho \times D$ the moment of the force HZ, taken with respect to the same axis, or the sum of the moments of all the forces which act upon the body, taken with respect to this same axis. That is, the centre of gravity will move parallel to the direction of the force ϱ, with a velocity $= \frac{\varrho}{L}$, L being the mass of the body; and if we draw GZ perpendicular to HZ, and call v the velocity of rotation of the point Z, we shall have Fig. 185. 124. 354, &c. 28.

$$v = \frac{\varrho \times GZ}{\int m\,r^2} \times GZ, \qquad 362.$$

or

$$v = \frac{\varrho \times \overline{GZ}^2}{\int m\,r^2}.$$

Of this we shall give a few applications.

380. Let us suppose that the body N impinges upon the body L, according to any direction whatever EQ, in such a man- Fig. 186.

ner as to cause no rotation in L, except about a single axis perpendicular to the plane which passes through the centre of gravity G, and the perpendicular TZ belonging to the point of contact T; it is proposed to determine the velocities after collision, and the directions of these velocities, the body L being supposed at rest.

Let us imagine a plane touching the point T, and let the velocity of N, according to EQ be decomposed into two others one according to ET perpendicular to this plane, and the other according to EI parallel to this same plane. If N had no other velocity but EI, it would only touch L in passing, and would communicate to it no motion, the effect of friction being out of the question. It is therefore only in virtue of the velocity ET, that the impulse is produced. Now as it is easy to determine ET in the parallelogram $ETAI$, of which all the angles and the diagonal EA are supposed to be known, we shall consider this velocity ET as known and we shall call it u. Let u' represent the velocity of N after collision, according to the direction ET or CZ; consequently $u - u'$ is the velocity lost by collision, and $N \times (u - u')$ is the force impressed upon the body L, which we have called ϱ. Therefore the centre of gravity and all the parts of the body will move in the direction GM parallel to CZ, with a velocity

$$v = \frac{N \times (u - u')}{L} \qquad \text{(I)},$$

calling this velocity v.

But, as the force $N \times (u - u')$ does not pass through G, the centre of gravity of L, the body must turn about G, as if this
136. point were fixed. Let v' be the velocity of rotation of the point Z where GZ, perpendicular to CZ, meets the latter line; we shall have, therefore,

363. $$v' = \frac{N \times (u - u') \times \overline{GZ}^2}{\int m\, r^2},$$

or representing GZ by D,

$$v' = \frac{ND^2\,(u - u')}{\int m\, r^2} \qquad \text{(II)}.$$

It may be observed, moreover, that it is necessary, in order that the body N may really have the velocity u', that the point T of the body L should also have this same velocity u' according to TZ. Let us now see with what velocity this point must advance according to TZ.

It will have, in the first place, the velocity v common to all the parts of L. Moreover, if we suppose that the infinitely small arc TT' perpendicular to GT, represents the velocity of rotation of the point T, by constructing the parallelogram $TCT'B$ upon the directions TT', TA and TZ, we shall have TZ for the velocity of T according to TZ in virtue of its rotation. Now the similar triangles $TT'C$, GTZ, give

$$GT : GZ :: TT' : TC = \frac{GZ \times TT'}{GT}.$$

But since v' is the velocity of rotation of the point Z, we have

$$v' : TT' :: GZ : GT,$$

whence

$$TT' = \frac{v' \times GT}{GZ};$$

and consequently

$$TC = \frac{GZ}{GT} \times \frac{v' \times GT}{GZ} = v';$$

therefore the total velocity of the point T belonging to the body L, according to EZ, is $v + v'$; and hence $v + v' = u'$ (III).

If from the three equations found above, in order to express the conditions of the motion, we deduce the values of u', v', and v, we shall obtain

$$u' = \frac{N\,(\int m\,r^2 + LD^2)\,u}{(N+L)\int m\,r^2 + LD^2\,N},$$

$$v = \frac{N\,u \int m\,r^2}{(N+L)\int m\,r^2 + LD^2\,N},$$

$$v' = \frac{LND^2\,u}{(N+L)\int m\,r^2 + LD^2\,N}.$$

If the distance GZ or $D = 0$; that is, if the direction of the impulse passes through the centre of gravity G, the velocity of rotation $v' = 0$, the velocities u' and v are equal to each other and to $\frac{N u}{N + L}$, as indeed they ought to be, according to article 288. The velocity u' being determined, if it be compounded with the velocity EI, which has suffered no alteration, we shall have the absolute velocity of N, and its direction after collision.

If the body L were in motion before collision, we should decompose the velocity of N before collision into two others, one of which should be equal and parallel to that of L; this would contribute nothing to the impulse, and we should employ the second as we have employed the velocity according to EQ, considering the body L as at rest.

If we compare the value found above for v' with that which we
363. before found for the velocity of rotation, by attending to the difference in the import of r in the two cases, we shall be able to determine the difference between the velocity of rotation which belongs to a free body, and that belonging to one which admits only of a rotation about a determinate point or axis.

381. From the value which we have found for v', the velocity of rotation, may be deduced a method for determining by experiment the value of $\int m r^2$, and the position of the centre of gravity in a body of any figure whatever. We shall apply to a vessel what we have to say upon this subject.

Let us suppose, that, by means of a weight N and a rope attached near the stern, the vessel is drawn in a direction perpendicular to its length, the weight being small compared with the whole weight of the vessel. Let this weight pass, for instance, over the
Fig.187. pulley P. The velocity of the vessel during the experiment (which should continue only for a very short time, as a minute or half a minute,) will be so small as to make it unnecessary to take account of the resistance of the water.

The action of gravity communicates to N, in the instant $d\,t$
273. the velocity $g\,d\,t$ (g being the velocity acquired in a second of time), and produces in the vessel an infinitely small velocity of rotation, which I shall call $d\,v'$ for the point A where the rope is

attached. Putting, therefore, $g\,d\,t$ for u, and $d\,v'$ for v', in the value of v' found above, N being considered as very small or nothing compared with L, the mass of the vessel, which gives

$$v' = \frac{ND^2\ u}{\int m\ r^2},$$

we shall have

$$d\,v' = \frac{g\ ND^2\ d\,t}{\int m\ r^2}.$$

Let $d\,v''$ be the velocity with which that point of the vessel turns which is distant one foot from the centre of gravity; we shall have

$$d\,v' : d\,v'' :: AG : 1 :: D : 1,$$

and consequently

$$d\,v' = D\ d\ v''.$$

Substituting for $d\,v'$ this value, we have

$$d\,v'' = \frac{g\ ND\ d\,t}{\int m\ r^2},$$

and, by integrating,

$$v'' = \frac{g\ ND\ t}{\int m\ r^2}.$$

Let z be the arc described by the point in question during the 280.
time t; we shall have

$$d\,z = v''\,d\,t,$$

and consequently

$$d\,z = \frac{g\ ND\ t\ d\,t}{\int m\,r^2};$$

whence, by integrating,

$$z = \frac{g\ ND\ t^2}{2\int m\ r^2}.$$

Therefore, if the rope, acting always perpendicularly to the length of the vessel, be attached to another point I, and we call z' the arc described by the same point during the same time t, we shall have

$$z' = \frac{g\,ND'\,t^2}{2\int m\,r^2},$$

calling D' the distance IG; whence we have

$$z : z' :: D : D' :: AG : IG;$$

Alg.224. and hence

$$z - z' : z :: AG - IG \text{ or } AI : AG.$$

Now if after each experiment we measure, as may easily be done in several ways, the angles of rotation, that is, the number of degrees contained in the arcs z, z' respectively, we may substitute these numbers instead of the arcs z, z', in the proportion; and since the distance AI is known, we readily obtain AG, that is, the position of the centre of gravity.

Geom. 294. The value of AG or D being determined, we calculate the length of the arc z which has 1 for radius, and of which the number of degrees is known; then, since N is known, and g is
271. equal to 32,2 feet; if we take care to observe the number of seconds which elapse up to the instant at which the number of degrees in z is counted, we shall know every thing except $\int m\,r^2$ in the equation

$$z = \frac{g\,ND\,t^2}{2\int m\,r^2};$$

but this equation gives

$$\int m\,r^2 = \frac{g\,ND\,t^2}{2\,z};$$

whence we obtain the value of $\int m\,r^2$, which it would be very troublesome to obtain by a particular calculation of the different parts of the vessel.

Fig.188. 382. When a body L of any figure whatever, having received an impulse in a direction HZ, not passing through the centre of gravity, takes the two motions of which we have spoken, it is easy to see, that for an instant it may be regarded as having but one single motion, namely, a motion of rotation about a fixed point or axis F, which according to the figure of the body, and also the distance GZ at which the impulse passes from G, may be situated either within or without the body. For if, while the

line GZ is carried parallel to itself from GZ to $G'Z'$, we imagine it to turn about the movable point G, since the points of the body have velocities of rotation greater in proportion to their distances from G, it is manifest that there is upon the line ZG a point F which will be found to have described from F' toward F, an arc equal to GG', and which may be regarded for an instant as a straight line; the point F then will have retrograded as far by its motion of rotation as it has advanced by the velocity common to all parts of the body; this point will therefore have remained constantly in F, which, for this reason, may be considered for an instant, as a fixed point about which the body turns. If we would know the position of the point F, it will be remarked that the arcs FF', $Z'I$, which the points F' and Z' describe in an instant, may be considered as straight lines perpendicular to GZ, or parallel to GG'; now the similar triangles $FF'G'$, $G'Z'I$, give

$$G'Z' : G'F' :: Z'I : FF',$$

or

$$GZ : GF :: Z'I : GG';$$

but we have found the velocity,

$$GG' = \frac{\varrho}{L}, \text{ and the velocity } Z'I = \frac{\varrho \times D^2}{\int m\, r^2};$$

hence,

$$GZ \text{ or } D : GF :: \frac{\varrho \times D^2}{\int m\, r^2} : \frac{\varrho}{L};$$

therefore

$$GF = \frac{\int m\, r^2}{D \times L}.$$

383. The point F is called the *centre of spontaneous rotation*, because it is a centre which the body takes as it were of itself. This point is precisely the centre of oscillation which the body L would have, if it turned about a fixed point or axis situated in Z; for from

$$GF = \frac{\int m\, r^2}{D \times L},$$

we have

$$FZ = \frac{\int m\, r^2}{D \times L} + GZ = \frac{\int m\, r^2 + L \times GZ \times D}{D \times L}$$

$$= \frac{\int m\, r^2 + L \times \overline{GZ}^2}{GZ \times L}.$$

Now $\int m\, r^2 + L \times \overline{GZ}^2$ is in article 360 precisely what we have understood by $\int m\, r^2$ in article 361; therefore the point F is here the same as the point O in article 361.

We perceive, therefore, that the point about which a body may be considered as turning for an instant, is independent of the value of the force or forces which are applied to this body; and generally it may be inferred from the value of FG, that this point is the more distant, according as the force in question, or the resultant of all the forces, acts at a less distance from the centre of gravity.

361. 384. We have seen that when a body turns about a fixed point or axis, its centre of percussion is the same as its centre of oscillation; whence these two centres are found by the same operation. It is not the same when the body is free. For, let us suppose a body whose mass is L, to turn about its centre of gravity with a velocity, which, for a point situated at the known distance a, shall be v; and that at the same time this centre moves with the velocity u. It is manifest, in the first place, that the resulting force of all the motions belonging to the different parts
136. of this body, will have for its value $L \times u$ or $L\, u$, that is, the same as if the body had no motion of rotation. In the second place, the distance at which the resultant must pass from the centre of gravity, is evidently that at which a force equal to $L\, u$ would produce in the body a velocity of rotation equal to that
356. which it actually has; but this velocity v has for its expression $\frac{L\, u \times D \times a}{\int m\, r^2}$, calling D the distance sought; we have, therefore,

$$v = \frac{L\, u\, D\, a}{\int m\, r^2},$$

and consequently

$$D = \frac{v}{u} \times \frac{\int m\, r^2}{L\, a};$$

and hence we see that the distance of the centre of percussion of a free body depends on the ratio of the velocity of rotation to the velocity of the centre of gravity; and particularly that it is nothing when the velocity of rotation is nothing, as in fact it ought to be.

We may hence determine at what point to place an obstacle in order to stop a free body which has a progressive and rotatory motion at the same time; namely, at the centre of percussion of this body, or the point where it would give the strongest blow or exert the greatest force.

Method of estimating the Forces applied to Machines.

385. Any force has for its measure, as we have already said, the product of a determinate mass, into the velocity which the force in question is capable of giving to this mass. It seems proper, in this place, to add something by way of illustrating the application of this principle to machines.

When two weights act against each other by means of a simple fixed pulley, it is necessary in order to an equilibrium that their masses should be equal; and this equilibrium once established, will always remain.

But if instead of opposing a weight to a weight, we oppose the force of an animal, as that of a man, for example, although it be true that, in order to an equilibrium, this man has only to exert an effort equal to the weight to be sustained, that is, equal to the quantity of motion represented by the mass of this body multiplied into the velocity which gravity communicates in an instant; it is, nevertheless, evident that if the man were capable of but one such effort, the equilibrium would continue only for an instant, because gravity renews each successive instant the action which was destroyed in the preceding.

Mech. 34

It is not, therefore, by the mass only which the man supports, that we are to estimate his strength; but we must consider, also, the number of times that he is able to exert an action equal to that which gravity communicates every instant to the body. Now if g represents the velocity which gravity is capable of giving to a free body in a second of time; and $d\,t$ represents an in-
267. finitely small portion of any time t, $g\,d\,t$ will be the velocity which gravity gives during the instant $d\,t$, t being supposed to be reckoned in seconds. Therefore, if m be the mass which it is proposed to sustain, $m\,g\,d\,t$ will be its weight, or the quantity of
272. motion which gravity gives it each instant $d\,t$; it is accordingly the effort also which must be exerted each instant by the force which is to support m, either directly or by the aid of a pulley. Therefore, during any time t, this force must expend a quantity of motion equal to

$$\int m\,g\,d\,t \quad \text{or} \quad m\,g\,t.$$

Therefore, if t denotes the time at the end of which the agent is no longer able to support the mass m, $m\,g\,t$ may be regarded as the measure of his strength. We do not mean by this that he is no longer capable of exerting any effort; but his force having become unequal to the effect to be produced, it is considered as nothing with respect to this effect. Let us, for example, suppose that in order to support a weight of $50^{\text{lb.}}$ for an hour, it is proposed to employ a force, which acting by equal and infinitely small degrees, is known to produce in a mass of $20^{\text{lb.}}$, a velocity of 50 feet in a second, at the instant when this force is exhausted. It is manifest that this mass of $20^{\text{lb.}}$ will have a quantity of motion equal to

$$20^{\text{lb.}} \times 50 \text{ or } 1000.$$

Let us see, then, if this quantity of motion be equal to what the quantity $m\,g\,t$ becomes, by putting $50^{\text{lb.}}$ for m, an hour or $3600''$
273. for t, and 32,2 feet for g. It appears to fall far short of it; such a force, therefore, would not support a weight of $50^{\text{lb.}}$ during an hour. If we wished to know during what time, or what number of seconds, it would support it, we have only to suppose

$$m\,g\,t = 1000\,;$$

and, putting 50 for m, and 32,2 for g, we shall have

$$\frac{m\,g\,t}{m\,g} \text{ or } t = \frac{1000}{50 \times 32{,}2} = \frac{1000}{1610} = \frac{100}{161} = \frac{5''}{8} \text{ nearly};$$

that is, such a force would support a weight of $50^{\text{lb.}}$ only about $\frac{5}{8}$ of a second.

386. Let us now suppose that it is required not only to support the mass m during the time t, but also to move it during the same time with a uniform and known velocity v.

It is manifest that in communicating the velocity v, either successively or at once, to the body m, there must have been expended a quantity of motion equal to $m\,v$; and to maintain this velocity v during the time t, the action of gravity is to be resisted all the while just as if the body had remained at rest; that is, there must have been expended an additional quantity of motion equal to $m\,g\,t$; therefore to maintain in the mass m the velocity v during the time t, the agent must be capable of producing a quantity of motion equal to $m\,v + m\,g\,t$.

387. It is ascertained by actual trial, that a man can work at a machine like that represented in figure 97, for 8 hours successively, and cause the winch to make 30 turns a minute, the radius of the cylinder and that of the winch being each 14 inches, and the weight applied at the surface of the cylinder being $25^{\text{lb.}}$. This experiment determines the value of

$$m\,v + m\,g\,t,$$

and consequently the limit to be observed in estimating the force of a man working at a machine, and for a definite period of time. Indeed, since the radius of the winch and that of the cylinder are equal, the weight in this case passes through the same space with the power. Thus, the radius being 14 inches, at each turn the power passes through $28 \times 3{,}1416$, or 88 inches nearly; and Geom.
since it makes 30 turns a minute, it describes 44 inches a second, 294.
or $\frac{44}{12}$ of a foot; that is, the velocity

$$v = \frac{44}{12} = \frac{11}{3}.$$

The mass

$$m = 25^{\text{lb.}},\ g = 32{,}2^{\text{ft.}},\ \text{and}\ t = 8^{\text{h.}} = 28800''.$$

The substitutions being made, we have

$$m\,v + m\,g\,t = \tfrac{275}{3} + 23184000 = 23184092.$$

By means of this number, we can judge whether the strength of a man be sufficient to produce a proposed effect. For instance, if it be asked whether it be possible for a man, with the machine above referred to, to raise a weight of $60^{\text{lb.}}$ with a velocity of 10 feet in a second, during 6 hours, we shall perceive that it is not. For we should have in this case

$$m = 60^{\text{lb.}};\ v = 10;\ g = 32{,}2;\ t = 21600'';$$

which gives

$$m\,v + m\,g\,t = 600 + 41731200 = 41731800;$$

as this greatly exceeds 23184092, it follows that a single man is unequal to such an effect.

It may be remarked that in these two examples, the velocity v with which the man is supposed to move the weight, is of very little consequence in estimating the force required; for in the first example, the quantity of motion which answers to this velocity, is $\tfrac{275}{3}$; and in the second, 600; quantities which are very small compared with 23184092 and 41731800. Therefore, in the second example, if we are unable to produce the desired effect, it is not because the velocity is greater than in the first case, but chiefly because the mass and the time during which it is to be moved, require of the agent too great a quantity of motion.

While therefore the velocity required in the agent is small compared with $g\,t$, that is, with the velocity which a heavy body falling freely would acquire in the time during which the agent is supposed to be employed, we may take simply for the measure of the force in question, the quantity $m\,g\,t$; and we shall have

$$m\,g\,t = 23184000.$$

Thus, if the mass (the velocity with which it is to be moved being moderate) multiplied by the velocity which a heavy body falling freely would acquire in the time during which the power

is to act, forms a product less than the constant number 23184000, or exceeding it but a little, the power may be considered as sufficient for the proposed effect, it being supposed to act as in the two preceding examples. But if the velocity with which the weight is to be moved, is considerable compared with $g\ t$, it will be necessary to subtract from the constant number 23184092, the quantity of motion $m\ v$ due to the velocity with which the body is to be moved; and if the weight multiplied by $g\ t$, the velocity which a falling body would acquire in the time during which the machine is to be worked, forms a product smaller than the remainder above found, the power may be deemed sufficient.

388. In what we have now said, we have taken no account of friction. When the motion of the machine has become uniform, (which is the state in which machines ought to be considered) the effect of friction may be regarded as constant, and it may be compared to a new mass required to be moved together with the proposed mass. Thus, in the case above considered, the friction being supposed equivalent to the weight of a known part $\frac{a}{c}$ of the mass m, this resistance will require in the power a quantity of motion equal to $\frac{a}{c}\, m\, g\, t$, and thus,

$$m\, v + \frac{a}{c}\, m\, g\, t + m\, g\, t,$$

or

$$m\, v + \left(\frac{a}{c} + 1\right) m\, g\, t$$

will be the measure of the moving force.

If then, in the experiment above referred to (the axle being supposed to have a radius much less than that of the cylinder), we suppose the friction to have been $\frac{1}{12}$ of the weight, $m\ v$ being neglected, as it may be in this case, we must augment the number 23184000 by its twelfth part; then the force of a man in similar circumstances may be represented by the number 25116000. We see, therefore, that to be able to estimate with sufficient accuracy the force of a man, we must previously ascertain the ratio of the force of friction to that of the weight, in the experiment employed, with the view of determining this force. Then if k

be the value derived from this experiment for $\left(\frac{a}{c} + 1\right) m\, g\, t$, we shall have,

$$\left(\frac{a}{c} + 1\right) m\, g\, t = k,$$

neglecting $m\, v$, when v is small compared with $g\, t$. This equation will enable us to judge for any other supposed value of $\frac{a}{c}$, whether the force of a man will be sufficient to move the weight m during the proposed time t.

389. In all that we have now said, we have regarded the agent as acting immediately upon the weight, and as deriving no advantage from local circumstances and the nature of the machine. We may often rely upon a much greater effect than the particular considerations now presented would lead us to expect. For instance, in the use of the pulley a man may add to his own proper force the weight of his body, or a large part of it. There are, moreover, many other circumstances of which he may avail himself, and other machines which admit of similar expedients. Frequently the motion is not continued, but takes place by starts, as in the pulley; and if there is a loss on this account, there is also this advantage, that the agent by intervals of rest is capable of exerting the same action for a longer time. We shall not dwell upon these details which it will be always easy to take into the account after all that has been said, especially if we proceed according to experiments in which care has been taken to distinguish the several causes on which the action of the moving force depends, and to note what belongs to each.

It is commonly said that a man can continue during about eight hours, to exert an effort equal to $25^{\text{lb.}}$. It will be seen from what precedes, that such a statement does not sufficiently determine the value of the force in question; besides, it is necessary, as we shall soon undertake to show, to have regard to the velocity with which the man acts; it is no less necessary to consider also the manner in which the action is applied, and many other circumstances which we cannot now stop to enumerate. It is proper, when circumstances vary, to proceed in our calculations upon new experiments made with reference to these circumstances.

390. Although we have considered that case only, in which the weight transmits all its resistance to the power, it is not less easy, after what has been said respecting the ratio of the weight to the power in each machine, to determine whether by the aid of a particular machine, a given power will produce a proposed effect. In the wheel and axle, for instance, if the radius of the cylinder be δ, and that of the wheel D; in order that the weight may move with the velocity v, it is necessary that the power should have a quantity of motion equal to $\frac{m\,v\,\delta}{D}$; and since in the time t, the action of gravity would give to the body m the quantity of motion $m\,g\,t$, the power in order to sustain this effort must have the force or quantity of motion $\frac{m\,g\,t\,\delta}{D}$; finally, if the friction is equivalent to the $\frac{a}{c}$ part of the weight, m being supposed to be applied at the distance δ, the power will require the additional quantity of motion $\frac{a}{c} \times \frac{m\,g\,t\,\delta}{D}$; thus, in order to determine whether the power be sufficient to move with the velocity v during the time t, the mass m, upon a wheel and axle of which the radius of the axle is δ, and that of the wheel D, we must determine by experiment the value of

$$\frac{m\,v\,\delta}{D} + \left(\frac{a}{c} + 1\right)\frac{m\,g\,t\,\delta}{D},$$

by employing at a wheel and axle, of known dimensions and known friction, a man moving a known mass; then if k is the value found by putting for m, v, δ, D, $\frac{a}{c}$, and t, the values of these quantities respectively in the experiment, it will be necessary, in every other case, that

$$\frac{m\,v\,\delta}{D} + \left(\frac{a}{c} + 1\right)\frac{m\,g\,t\,\delta}{D}$$

should have a value not exceeding k.

So also, upon the inclined plane, the power acting parallel to the plane; if we call i the inclination of the plane, $m\,g\,t \sin i$ 207.
will be the quantity of motion which gravity will communicate

successively to the movable body, according to the directions of the plane, in the time t; thus the power will be required to have a quantity of motion equal to

$$m v + m g t \sin i;$$

and if the friction be the $\frac{a}{c}$ part of the weight, it will be necessary to employ a quantity of motion equal to

$$m v + m g t \sin i + \frac{a}{c} m g t.$$

Having, therefore, determined by experiment one value of

$$m v + m g t \sin i + \frac{a}{c} m g t,$$

it will be necessary when we wish to determine whether the same power be capable of moving a given mass m, with a known velocity v, during a known time t, upon a plane whose inclination is i, and upon which the friction is a known part of the weight; it will be necessary, I say, to determine whether the value which

$$m v + m g t \sin i + \frac{a}{c} m g t$$

will then have, is less than that in the experiment, or at most only equal to it; in either case the thing will be possible.

If the time t, during which the machine is to be in motion, be not given; still if we know the space which the power or the weight must describe with the velocity v; then, as we suppose that the motion is uniform, if we call s the space which the weight is to
24. pass through, we should put instead of t its value $\frac{s}{v}$.

Such is, in substance, the method which is to be pursued in estimating forces applied to machines. Each machine requires particular considerations as to the nature of the power and the manner in which it is applied to this machine. But by going back to the quantity of motion to be expended by the agent, we may always determine whether he be capable of a proposed

effect; and the principles which we have now laid down, will serve to conduct us in such inquiries.

Of the Maximum Effect of Agents and Machines.

391. When any power is made to act upon a given resistance, by the intervention either of a simple or a compound machine, an equilibrium will take place when the velocity of the power is to the velocity of the resistance as the weight is to the power. In this state of things, however, the machine must be actually at 227.
rest, and therefore incapable of performing any work. If we can increase the power, the machine will move with more and more velocity, and will have its motion gradually accelerated as long as the power exceeds the resistance. But if from any cause the power should begin to diminish, or if the resistance should increase, or if both these changes in the state of the machine should take place at the same time, the acceleration of the machine will diminish, and it will at last arrive at a state of uniform motion. Now this increase of resistance may arise in many cases from an increase of friction, which often (though not always) accompanies an augmentation of velocity; or it may arise from the resistance of the air, which must necessarily increase with the velocity; and therefore all machines are found soon to attain a state of uniform motion. When an undershot wheel is driven by the impulse of water, the uniformity of motion to which it arrives, arises principally from the diminution of the power which in this case accompanies an increase of velocity. When the mass of fluid strikes one of the float-boards at rest, the impulse is then a maximum. When the float-board is in motion it withdraws itself, as it were, from the action of the power, and therefore its mechanical effect will diminish as the velocity increases, and if it were possible that the velocity of the wheel should become equal to that of the fluid, the float-board would not be struck at all by the moving water. Hence it follows, that the power itself diminishes by an increase of velocity, and therefore that from this cause alone machines in general would soon acquire a motion sensibly uniform. This effect will be more easily understood, if we suppose an axle to be put in motion by

two currents of water, moving with different velocities and driving two wheels, one of which is placed at each extremity of the axle. When the wheels have begun to move, by the joint action of these falls of water, its motion will at first be slow, and each fall of water will perform its part in giving motion to the axle; but if the greater fall is capable, by the continuance of its action, of giving its wheel a velocity either equal to, or greater than the velocity of the smaller fall, then it is manifest that the smaller fall ceases to impel its wheel, and that the whole effect is produced by the action of the greater fall. Hence it will be perceived from this statement, not only why a diminution of the impelling power accompanies an increase of velocity, but why there is a certain velocity of the machine, which is necessary before we can gain all the useful effect which we wish to have from the powers which we employ.

392. In order to illustrate this in the case of a real machine, let us suppose that the power of a man is to be employed in raising a load by means of a walking crane. This machine consists of a large wheel placed upon an axle, round which is coiled a rope, having a weight r attached to its lower extremity; the man walks upon the interior of the wheel, and by his weight gives it a rotatory motion, and thereby coils the rope round the axle, and elevates the weight r. Let us suppose the wheel or *drum* so constructed, like the fusee of a watch, that the man can walk at different distances from the axis; and let p be the power or weight of the man, r the weight to be raised, and δ the distance of the latter or radius of the axle, and D the distance of the former or the radius of the wheel; then

$$p : r :: \delta : \mathrm{D} = \frac{r\,\delta}{p},$$

the distance from the centre of the wheel, at which the man must place himself, in order to be in equilibrium with the resistance r. But as the machine must be moved, and the weight raised, the man must go to a greater distance from the axis than $\frac{r\,\delta}{p}$; the motion of the machine will therefore be accelerated, and the acceleration would increase as he moved to a greater and greater distance from the centre of the wheel. Hence it is obvious, that as the acceleration increases, the man must walk with greater and

greater velocity; but there is an obvious limit to this, for he would soon be fatigued by the rapid walking, and would therefore be rendered unfit to continue his work. He must therefore return to that distance from the axis, where the wheel has such a velocity that he can continue to move with that velocity during the period that his work is to last; that is, there is a particular velocity with which the man must walk, in order to perform the greatest quantity of work; and it would be easy to find this velocity, if we knew the law according to which his force is diminished, as his velocity increases. We may suppose, however, that his force diminishes in the same ratio as his velocity increases.

393. Let p represent the force which a man can exert during a given time against a dead weight. This force will obviously be greater than any which he would exert on the supposition of motion taking place; for a part of his strength in this case would be expended in putting himself in motion and in continuing this motion. Let v be the velocity with which he would lose the power of exerting any force; then, if he move with a velocity v' less than v, he will exert a force less than p, and the part lost may be found, according to the above hypothesis, that the diminution of force is as the increase of velocity. Since he loses all his force, or p, when the velocity is v, and none when there is no velocity, we have

$$v - 0 \text{ or } v : v' - 0 \text{ or } v' :: p : \frac{p\,v'}{v}.$$

$\frac{p\,v'}{v}$ is therefore the loss of force sustained on account of moving with the velocity v'. There will accordingly remain

$$p - \frac{p\,v'}{v} \text{ or } p\left(1 - \frac{v'}{v}\right)$$

as the effective force actually exerted against the weight. Now if D be the distance at which this force acts, r the resistance or weight raised, and δ the distance at which the resistance acts, and u its velocity; then, when the machine has attained a uniform motion, we shall have

$$p\left(1 - \frac{v'}{v}\right) D = r\,\delta.$$

But, since

$$\text{D} : \delta :: v' : u,$$

we have, by substitution,

$$p\left(1 - \frac{v'}{v}\right) v' = r\,u.$$

To find the maximum we put the differential of the first member equal to zero, v' being regarded as variable; we have thus

Cal. 45.
$$p\,d\,v' - \frac{2\,p\,v'\,d\,v'}{v} = 0,$$

or

$$v = 2\,v' \quad \text{and} \quad v' = \tfrac{1}{2}\,v.$$

Substituting $\frac{1}{2}\,v$ for v' in the above equation, we obtain

$$r\,u = \tfrac{1}{4}\,p\,v.$$

On the hypothesis, therefore, which we have assumed, the man will do most work when he moves with half his greatest velocity, and in this case the greatest effect will be $\frac{1}{4}\,p\,v$.

394. It appears, however, by direct experiments, that the force of a man diminishes as the square of his velocity increases,* in other words, that the effective forces are as the squares of the diminutions of velocity from the point where the effective force is nothing. Calling p', therefore, the force answering to the velocity v', we shall have, according to this hypothesis,

$$p : p' :: (v - 0)^2 : (v - v')^2;$$

whence

$$p' = p\left(\frac{v - v'}{v}\right)^2 = p\left(\frac{w}{v}\right)^2 \text{ (I), putting } v - v' = w \text{ (II)};$$

and hence,

$$p\left(\frac{w}{v}\right)^2 v' \text{ or } p\left(\frac{w}{v}\right)^2 (v - w) = r\,u,$$

since

$$v' = v - w.$$

Taking the differential, as before, and putting it equal to zero, and suppressing the constant factor, we have

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* See note on the measure of forces.

$$2\,v\,w\,d\,w - 3\,w^2\,d\,w = 0\,;$$

whence $2\,v = 3\,w$, and $w = \frac{2}{3}\,v$. Substituting this value in equations (II), (I), we obtain

$$v - v' = \frac{2}{3}\,v,$$

or

$$v' = v - \frac{2}{3}\,v = \frac{1}{3}\,v\,;$$

also

$$p' = p\left(\frac{\frac{2}{3}\,v}{v}\right)^2 = \frac{4}{9}\,p\,;$$

that is, *the work done is a maximum when the agent moves with one third part of the greatest velocity of which he is capable, and when the weight or load is* $\frac{4}{9}$ *of the greatest which he is able to put in motion* during the whole time he is supposed to act.

395. Having thus considered the maximum effect of living agents, we shall proceed to the subject of machines, and shall take the case of a wheel and axle, as almost all other machines may be reduced to this.

The powers by which a machine is put in motion, and by which that motion is kept up, are called *first movers*, or *moving powers*, or more familiarly, *mechanical agents;* and when various moving powers are applied to the same machine, the resultant of them, or the equivalent force, is called the *moving force.*

The first movers of machinery, are, the force of men and that of other animals, the force of steam, the force of wind, the force of moving water, the weight of water, the reaction of water, the descent of a weight, the elasticity of a spring, &c. If a machine be driven by two powers acting in two different directions, we must then find their resultant, and consider the machine as driven by the resulting force.

The powers which oppose the production of motion in a machine, and its continuance, are called *resistances ;* and the resultant of all the resisting forces is called the *risistance.*

The work to be performed is, in general, the principal resistance to be overcome ; but, in addition to this, we must consider

the resistance of friction, and the resistance arising from the inertia of all the parts of the machinery; for a certain portion of the moving power is necessarily wasted in overcoming these obstacles to motion.

The *impelled point* of a machine is that point at which the moving power is applied, or rather that point at which the moving force is supposed to act, when this moving force is the resultant of various powers differently applied. The *working point* of a machine is that point at which the resistance is overcome, or that point at which the resultant of all the resisting forces is supposed to act.

The *work performed*, or the *effect* of a machine, is equal to the resistance multiplied by the velocity of the working point.

The moment of impulse is equal to the moving force multiplied by the velocity of the impelled point.

396. In proceeding to investigate general expressions for the ratio of the velocities of the impelled and working points of machines, when their performance is a maximum, let

$\mathrm{D} =$ the radius of the wheel to which the power is applied; or, which is the same thing, the velocity of the impelled point of the machine;

$\delta =$ the radius of the axle to which the resistance is applied, or the velocity of the working point of the machine;

$p =$ the moving force applied at the impelled point;

$r =$ the resistance arising solely from the work to be performed;

$m =$ the inertia of the moving power p, or the quantity of matter to which that power must communicate the velocity of the impelled point;

$n =$ the inertia of the resistance, or the quantity of matter to be moved with the velocity of the working point, before any work can be performed;

$f =$ the quantity of matter, which, if placed at the working point, would create the same resistance as friction;

$i =$ the quantity of matter, which if placed at the working point, would oppose the same resistance as the *inertia* of all the parts of the machinery.

Since D and δ are the radii of the wheel and axle, we shall have $D : \delta :: r : \frac{r\,\delta}{D}$, a weight equal to that part of the power p which is in equilibrium with the resistance. We have, therefore, $p - \frac{r\,\delta}{D}$ as an expression for the effective force of the power; and as D is the distance at which this force is applied, we have

$$p\,D - r\,\delta$$

to represent the force which is employed in giving a rotatory motion to the machine. The resistance which friction opposes to this force will be $f\,\delta$; the moment of inertia of the power p will be as $m\,D^2$; the moment of inertia of the resistance as $n\,\delta^2$, and the moment of inertia of the machinery will be as $i\,\delta^2$. Since the moving force is diminished by the resistance of friction, we shall have $p\,D - r\,\delta - f\,\delta$ for the moving force; and since the resistance arises from the moment of inertia of the resistance, the moment of inertia of the power, and that of the machinery, it will be as $m\,D^2 + n\,\delta^2 + i\,\delta^2$. But the velocity is proportional to the moving force directly and to the resistance inversely; therefore

the rotatory velocity will be

$$\frac{p\,D - r\,\delta - f\,\delta}{m\,D^2 + n\,\delta^2 + i\,\delta^2}.$$

Now, since the velocities of the impelled and working points are as their distances from the centre of motion, or as D and δ, we shall obtain these velocities respectively by multiplying the rotatory velocity by D and δ; and as the work performed is equal to the resistance multiplied by the velocity of the working point;

we shall have for the velocity of the impelled point

$$\frac{p\,D^2 - r\,D\,\delta - f\,D\,\delta}{m\,D^2 + n\,\delta^2 + i\,\delta^2};$$

for the velocity of the working point

$$\frac{p\,\mathrm{D}\,\delta - r\,\delta^2 - f\,\delta^2}{m\,\mathrm{D}^2 + n\,\delta^2 + i\,\delta^2};$$

and for the work performed

$$\frac{r\,p\,\mathrm{D}\,\delta - r^2\,\delta^2 - r\,f\,\delta^2}{m\,\mathrm{D}^2 + n\,\delta^2 + i\,\delta^2}.$$

In order to obtain absolute measures of the velocities and the work performed, we must consider that, q being the accelerating force, and $q\,g$ the velocity acquired in a second, we shall have $1 : t :: q\,g : v = q\,g\,t$; and as the accelerating forces are proportional to the velocities generated by them in equal times, the preceding expressions for the velocities of the impelled and working points may be substituted for the accelerating force q in the equation $v = q\,g\,t$, and we shall obtain, for the absolute velocity of the impelled point

$$\frac{p\,\mathrm{D}^2 - r\,\mathrm{D}\,\delta - f\,\mathrm{D}\,\delta}{m\,\mathrm{D}^2 + n\,\delta^2 + i\,\delta^2} \times g\,t;$$

for the absolute velocity of the working point,

$$\frac{p\,\mathrm{D}\,\delta - r\,\delta^2 - f\,\delta^2}{m\,\mathrm{D}^2 + n\,\delta^2 + i\,\delta^2} \times g\,t;$$

and for the work performed

$$\frac{r\,p\,\mathrm{D}\,\delta - r^2\,\delta^2 - r\,f\,\delta^2}{m\,\mathrm{D}^2 + n\,\delta^2 + i\,\delta^2} \times g\,t.$$

This is a maximum when the differential, δ being considered as variable, is equal to zero, which gives

$$\begin{aligned}&(p\,\mathrm{D} - 2\,\delta\,(r+f))\,(m\,\mathrm{D}^2 + \delta^2\,(n+i))\\ &\quad - 2\,\delta\,(n+i)\,(p\,\mathrm{D}\,\delta - \delta^2\,(r+f)) \qquad = 0,\end{aligned}$$

or, by reducing,

$$p\,m\,\mathrm{D}^3 - p\,\mathrm{D}\,\delta^2\,(n+i) - 2\,\delta\,m\,\mathrm{D}^2(r+f) \qquad = 0;$$

that is,

$$\delta^2 + \frac{2\,m\,\mathrm{D}\,(r+f)\,\delta}{p\,(n+i)} = \frac{m\,\mathrm{D}^2}{n+i}.$$

Resolving this after the manner of an equation of the second degree, we obtain

$$\delta = \text{D}\,\frac{\sqrt{m^2\,(r+f)^2 + p^2\,m\,(n+i)} - m\,(r+f)}{p\,(n+i)}.$$

When $r = 0$, we have

$$\delta = \text{D}\,\frac{\sqrt{m^2 f^2 + p^2\,m\,(n+i)} - m f}{p\,(n+i)}.$$

This case takes place when the resistance to be overcome exerts a contrary strain on the machine, while it consists merely in the inertia of the impelled body; as in driving a millstone, a fly, or in pushing a body along a horizontal plane.

When $f = 0$,

$$\delta = \text{D}\,\frac{\sqrt{m^2\,r^2 + p^2\,m\,(n+i)} - m\,r}{p\,(n+i)}.$$

This case takes place when the friction is so small that it may be disregarded, which often happens in good wheel-work, where the surfaces that touch one another are very small.

When $r = 0$, and $f = 0$, we have

$$\delta = \text{D}\,\frac{\sqrt{p^2\,m\,(n+i)}}{p\,(n+i)} = \text{D}\,\sqrt{\frac{p^2\,m\,(n+i)}{p^2\,(n+i)^2}} = \text{D}\,\sqrt{\frac{m}{n+i}}.$$

This case takes place when the circumstances of the two preceding cases are combined.

When $n = 0$, we have

$$\delta = \text{D}\,\frac{\sqrt{m^2\,(r+f)^2 + p^2\,m\,i} - m\,(r+f)}{p\,i}.$$

This case takes place in the grinding of corn, the sawing of wood, the boring of wooden or iron cylinders, &c., where the quantity of motion communicated to the flour, the saw-dust, or the iron filings, is too trifling to be taken into the account.

When $r = 0$, $f = 0$, and $n = 0$, we have $\delta = \text{D}\sqrt{\frac{m}{i}}$.

When $m : n :: p : r$, we have,

$$\delta = \text{D}\,\frac{\sqrt{p^2 (r+f)^2 + p^3 (r+i)} - p\,(r+f)}{p\,(r+i)}.$$

This case takes place when the inertia of the power and the resistance are proportional to their pressure; as when water, minerals, or any other heavy body, is raised by means of water acting by its weight in the buckets of an overshot wheel.

When, in the last case, $i = 0$, and $f = 0$, we have

$$\delta = \text{D}\sqrt{\frac{p^2 (r+f)^2 + p^3 (r+i)}{p^2 (r+i)^2}} - \text{D} = \text{D}\sqrt{\frac{p}{r} + 1} - \text{D}.$$

This case often takes place, and particularly in pulleys; and making $\text{D} = 1$, and $r = 1$, we obtain

$$\delta = \sqrt{p + 1} - 1;$$

and when $p = 1$, and $\text{D} = 1$, we have

$$\delta = \sqrt{\frac{1}{r} + 1} - 1.$$

The preceding formulas will be found applicable to almost every case which can occur; and the intelligent engineer will have no difficulty in accommodating them to any unforeseen circumstances.

The following table will, in many cases, save the trouble of calculation. It is computed from the formula

$$\delta = \text{D}\sqrt{\frac{p}{r} + 1} - \text{D},$$

D being supposed $= 1$, and $r = 10$.

Table containing the best Proportions between the Velocities of the Impelled and Working Points of a Machine, or between the Levers by which the Power and Resistance act.

Proportional value of the impelling power, or p.	Value of the velocities of the working point, or δ, or of the lever, by which the resistance acts, that of D being 1.	Proportional value of the impelling power, or p.	Value of the velocities of the working point, or δ, or of the lever, by which the resistance acts, that of D being 1.
1	0.048809	20	0.732051
2	0.095445	21	0.760682
3	0.140175	22	0.788854
4	0.183216	23	0.816590
5	0.224745	24	0.843900
6	0.264911	25	0.870800
7	0.303841	26	0.897300
8	0.341641	27	0.923500
9	0.378405	28	0.949400
10	0.414214	29	0.974800
11	0.449138	30	1.000000
12	0.483240	40	1.236200
13	0.516575	50	1.449500
14	0.549193	60	1.645700
15	0.581139	70	1.828400
16	0.612451	80	2.000000
17	0.643168	90	2.162300
18	0.673320	100	2.316600
19	0.702938		

In order to understand the method of using this table, let us suppose that we wish to raise two cubic feet of water in a second, by means of the power of a stream which affords five cubic feet of water in a second, applied to a wheel and axle, the diameter of the wheel being seven feet. It is required, therefore, to find the diameter which we must give to the axle, in order to obtain a maximum effect. We have obviously $p = 5$, and $r = 2$, and since $p : r :: 5 : 2$, we have $p = \frac{5}{2} r$; but, in the above table, $r = 10$; hence $p = \frac{5}{2} 10 = 25$. Now it appears from the table, that when $p = 25$, the diameter of the axle, or δ, is 0.8708, D being 1; but as D $= 7$, the diameter of the axle must be $7 \times 0.8708 = 6.0956$.

397. When a machine is already constructed, the velocity of its impelled and working points are determined; and therefore in order to obtain from it its maximum effect, we must seek for the best proportion between the power and the resistance, as these are the only circumstances over which we have any control, without altering the machinery.

In order to find the ratio of p to r, which would produce a maximum effect, it is requisite only to make r variable in the formula above given; but it often happens, that when r varies, the mass n suffers a considerable change, although there are other cases when the change in n is too inconsiderable to be noticed.

Let us, therefore, first take the case when r alone varies without inducing a change in n. In this case, the expression for the work performed, namely,

$$\frac{r\,p\,\mathrm{D}\,\delta - r^2\,\delta^2 - r\,f\,\delta^2}{m\,\mathrm{D}^2 + n\,\delta^2 + i\,\delta^2},$$

will be a maximum when

$$r = \frac{p\,\mathrm{D} - f\,\delta}{2\,\delta};$$

as will be readily found by differentiating, putting the differential equal to zero, and deducing the value of r. But according to the experiments of Coulomb, the friction is in general equal to $\frac{1}{15}$th of the resisting pressure. Hence we may omit $f\,\delta$, and consider the resistance as $= r + \frac{1}{15}\,r = \frac{16}{15}\,r$. Consequently, $\frac{16}{15}\,r = \frac{p\,\mathrm{D}}{2\,\delta}$, and $r = \left(\frac{p\,\mathrm{D}}{2\,\delta}\right) \times \frac{15}{16}$. But if we consider the fraction $\frac{15}{16}$ as so near 1, that the substitution of the latter will not greatly affect the result, we shall obtain, by making $p = 1$ and $\mathrm{D} = 1$, $r = \frac{1}{2\,\delta}$; that is, the resistance should be nearly one half of the force which would keep the impelling power in equilibrium, a rule which is applicable to many cases where the matter moved by the working point of the machine is inconsiderable.

398. In those cases where n varies along with r, it will in general vary in the same proportion, and we may therefore represent n by $x\,r$, some multiple of r. For the sake of simplicity, the friction f may be considered as absorbing a certain portion of the impelling power, which will then be represented by $p - f$; and we may also regard the inertia of the machine, or i, as applied at the impelled instead of the working point; that is, the moment of inertia may be considered as proportional to $i\,\mathrm{D}^2$. Now, if we make $p - f = 1$, and $\mathrm{D} = 1$, in the formula

$$\frac{r\,p\,\mathrm{D}\,\delta - r^2\,\delta^2 - r\,f\,\delta^2}{m\,\mathrm{D}^2 + n\,\delta^2 + i\,\delta^2},$$

we shall obtain

$$\frac{r\,\delta - r^2\,\delta^2}{m + i + x\,r\,\delta^2};$$

and making $m + i = s$, we have

$$\frac{r\,\delta - r^2\,\delta^2}{s + x\,r\,\delta^2}$$

for the work performed.

This is a maximum when the differential, r being considered as variable, is equal to zero, which gives

$$(\delta - 2\,r\,\delta^2)\ (s + x\,r\,\delta^2) - x\,\delta^2\,(r\,\delta - r^2\,\delta^2) = 0;$$

or, by reducing,

$$s\,\delta - 2\,s\,r\,\delta^2 - x\,r^2\,\delta^4 = 0;$$

that is,

$$r^2 + \frac{2\,s\,r}{x\,\delta^2} = \frac{s}{x\,\delta^3},$$

and by resolving this after the manner of an equation of the second degree, we obtain

$$r = -\frac{s}{x\,\delta^2} + \sqrt{\frac{s}{x\,\delta^3} + \frac{s^2}{x^2\,\delta^4}} = \frac{\sqrt{s\,x\,\delta + s^2} - s}{x\,\delta^2}.$$

When $x = 1$, we have

$$r = \frac{\sqrt{s\,\delta + s^2} - s}{\delta^2}.$$

This case takes place when the machine is employed in raising a weight, drawing water, &c.

When $i = 0, f = 0$, and $m = p$, then $m + i$ or $s = p = 1$, and

$$r = \frac{\sqrt{\delta + 1} - 1}{\delta^2}.$$

When $D = \delta$, as in the common pulley, then $\delta = 1$, and

$$r = \frac{\sqrt{1 + 1} - 1}{1} = \sqrt{2} - 1 = 0.4142.$$

In order to save the trouble of calculation, we have added the following table, computed from the formula

$$r = \frac{\sqrt{\delta + 1} - 1}{\delta^2}.$$

Table containing the best Proportions between the Power and Resistance, the Inertia of the impelling Power being the same with its Pressure, and the Friction and Inertia of the Machine being omitted.

Values of δ, or the velocity of the working point, D being equal to 1.	Values of r, or the resistance to be overcome, p being $= 1$.	Ratio of r to the resistance which would balance p.	Values of δ, or the velocity of the working point, D being equal to 1.	Values of r, or the resistance to be overcome, p being $= 1$.	Ratio of r to the resistance which would balance p.
$\frac{1}{4}$	1.8885	0.4724 to 1	7	0.03731	0.26117 to 1
$\frac{1}{3}$	1.3928	0.4639 ——	8	0.03125	0.25000 ——
$\frac{1}{2}$	0.8986	0.4493 ——	9	0.02669	0.24021 ——
1	0.4142	0.4142 ——	10	0.02317	0.23170 ——
2	0.1830	0.3660 ——	11	0.02037	0.22407 ——
3	0.1111	0.3333 ——	12	0.01809	0.21708 ——
4	0.0772	0.3088 ——	13	0.01622	0.21086 ——
5	0.0580	0.2900 ——	14	0.01466	0.20524 ——
6	0.0457	0.2742 ——	15	0.01333	0.19995 ——

In order to understand the method of using this table, let us suppose that it is required to find the value of the resistance, or the quantity of water which must be put into a bucket to be raised by a wheel and axle, in which the radius of the wheel is 6 feet, and that of the axle 2 feet, and with a power $= 8$. Since, in the table, $D = 1$, we have

$$6 : 2 :: \text{D} = 1 : \delta = \frac{2}{6} = \frac{1}{3},$$

which corresponds in the table to 1,3928, the value of r when $p = 1$. But, in the present case, $p = 8$, consequently

$$1 : 8 :: 1.3928 : 11.1424,$$

the value of r when $p = 8$.

399. The subject of the maximum effect of machines may be considered in a very simple point of view, if we suppose, what is by no means improbable, that the moving power in machinery observes the same law that has been found to exist with regard to animal force, and also with regard to the force of fluids in motion. Upon this hypothesis, we shall have $r = p\left(1 - \frac{v'}{v}\right)^2$, and the effect of the machine will be $r\,v' = p\,v'\left(1 - \frac{v'}{v}\right)^2$, which will be a maximum when $\frac{r\,\delta}{\text{D}} = \frac{4}{9}\,p$, and when $v' = \frac{1}{3}\,v$. 394.
In these formulas, p is the load that is just sufficient to bring the machine to rest, or prevent it from moving, v is the greatest velocity of the power when no work is done, and v' the velocity of the impelled point of the machine. The above equation, both members being multiplied by 9, is equivalent to the proportion, D is to δ, or the velocity of the impelled point is to the velocity of the working point, when the effect is a maximum, as $9\,r$ to $4\,p$.

Table of the Strength of Men, according to different Authors.

Number of pounds raised.	Height to which the weight is raised.	Time in which it is raised.	Duration of the Work.	Names of the Authors.
1000	180 feet	60 minutes		Euler
60 French	1 French	1 second	8 hours	Bernoulli
25 French	220 French	145 seconds		Amontons
170 French	1 French	1 second	half an hour	Coulomb
1000	330	60 minutes		Desaguliers
1000	225	60 minutes		Smeaton
30	$3\frac{1}{2}$	1 second	10 hours	Emerson
30	2,43 feet	1 second		Schulze

The following are the estimates that have been made of the relative strength of horses, asses, and men.

1 horse	is equal to	5 men,	Desaguliers. Smeaton.
1	"	"	7 men,	Bossut, &c.
1 ass		"	2 men,	Bossut.

HYDROSTATICS.

INTRODUCTORY REMARKS.

400. Hydrostatics is that part of Mechanics which treats of the equilibrium of fluids, and that of solids immersed in them.

A *fluid* is a collection of material particles, so constituted as to yield to the smallest force employed to separate them. The fluids with which nature presents us, approach more or less to this state of perfect fluidity. The adhesion which exists among the particles of several of these substances, and which gives rise to what is called *viscidity*, opposes itself to the separation of the particles; but in the theory which we are about to unfold, no account is taken of this adhesion, and we have reference only to the perfect fluids.

We distinguish two kinds of fluids; the one *incompressible*; or nearly so,* as water, mercury, alcohol, and liquids generally. These are capable of taking an infinite variety of forms without any sensible change of bulk. The second kind of fluids comprehends atmospheric air, the gases generally, and vapors. These are in an eminent degree *compressible;* they are also endued with a perfect elasticity, and are thus capable of changing at the same time their form and bulk, upon being compressed, and of recovering their figure again when the compressing force is removed. Vapors differ from air and the gases, by losing the form of elastic fluids and returning to the state of liquids, when compressed to a certain degree, or when their

* See note subjoined to this treatise on the compressibility of water.

temperature is sufficiently reduced; whereas air and the gases are found, with few exceptions, to preserve always their elastic form in the state of the greatest compression and lowest temperature to which they have hitherto been reduced.*

401. Although we are unable to assign the magnitude of the elementary parts of fluid bodies, we cannot doubt that these parts are material, and that the general laws of equilibrium and motion, already established, are applicable to them as well as to solids. But as this law of equilibrium is not the only one required, some other is to be sought on which the equilibrium depends.

402. As an equilibrium consists in destroying all the forces employed, and as we do not know how the parts of a fluid transmit their forces among themselves, it is only by having recourse to experiment, that we are able to establish our first principles. We begin, therefore, by stating what is most clearly and certainly known upon this subject.

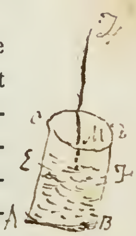
Pressure of Fluids.

Fig.189, 190. 403. Let $ABCD$ be a canal or tube, composed of three branches AB, BC, CD, of equal diameters. Let us suppose that a heavy fluid is poured into the branch AB; it will pass through the branch BC into the branch CD; and when we cease pouring, the surface of the fluid in the two branches will be in the same horizontal line, whatever be the inclination of the branch BC. This is a fact abundantly established and universally admitted. We proceed to make known the consequences to be deduced from it.

404. If through any point E, taken at pleasure, we imagine a horizontal line EF to pass, it is evident that the weight of the fluid $EBCF$ contributes nothing to the support of the columns AE, DF; and that consequently the equilibrium would still be preserved, if the fluid contained in $EBCF$ were suddenly deprived of its gravity. This fluid, therefore, is to be regarded

* See note on the condensation of gases into liquids.

simply as a medium of communication between the columns AE and DF; so that $EBCF$ transmits to the column DF all the pressure it receives from AE; and reciprocally it transmits to AE all it receives from DF. It is also evident that we should arrive at the same conclusions, if instead of the columns AE, DF, we substitute two forces of the same value; hence, as the result is not affected by any inclination of the branch BC, we conclude that, *if a fluid, destitute of gravity, be contained in any vessel, and* Fig.198. *if, having made an opening in the vessel, we apply to this opening any given pressure, the force thus exerted will diffuse itself equally in all directions.*



405. Now it will be readily seen, not only that the pressure transmits itself equally in all directions, but also that it acts at each point perpendicularly to the surface of the vessel containing the fluid; for if, acting on the surface, it did not act perpendicularly, its effect could not be entirely destroyed by the resistance of this surface; there would result therefore an action upon the parts of the fluid itself, which, as it could not but trans- 404. mit itself in all directions, would necessarily occasion a motion in the fluid; it would be impossible, therefore, on such a supposition, for a fluid to remain at rest in a vessel, which is contrary to experience.

406. We hence conclude, that if the parts of a fluid contain- Fig.191. ed in any vessel $ABCD$, open toward AD, are urged by any forces whatever, and are notwithstanding preserved in a state of equilibrium, these forces must be perpendicular to the surface AD; for if there be an equilibrium, this equilibrium would still obtain, if a covering were applied of the same figure with the surface AD; but we have just seen, that in this case the forces acting at the surface AD must be perpendicular to this surface.

407. Accordingly, let us suppose that the forces acting on the parts of the fluid are gravity itself; we shall infer that the direction of gravity is necessarily perpendicular to the surface of tranquil fluids; and that consequently *the parts of the same heavy fluid must be on a level, in order to be in equilibrium, whatever be the figure of the vessel.*

Fig.192. 408. Let us now suppose that the vessel $ABCD$, being closed on all sides, is filled with a fluid destitute of gravity, and that, having a very small opening at E, we apply to it any force; it is evident that the pressure that would hence be exerted upon the plane surface represented by BC, would not depend in any degree upon the quantity of fluid contained in the vessel, nor upon the figure of the vessel; but that, since the pressure applied
404. at E transmits itself equally in all directions, the pressure upon BC would be equal to that exerted upon any point of the opening E, repeated as many times as there are points in BC.

409. For the same reason, the pressure applied at E, transmitting itself in all directions, would tend to raise the superior surface AD, and the force thus exerted would be for each point equal to the pressure applied at any point of the opening E; so that the surface AD is pressed perpendicularly from within outward with a force equal to the pressure employed at any point of the opening E, repeated as many times as there are points in AD.

Fig. 193. 410. Let the vessel $ABCDEF$, the part CD being horizontal, be filled with a heavy fluid. We say that the pressure upon the bottom CD, arising from the gravity of the fluid, does not depend upon the quantity of fluid contained in the vessel, but simply upon the extent of CD, and its depth below the surface AF.

To make this evident, let us suppose, the line BE being horizontal, that the fluid contained in $BCDE$, is suddenly deprived of its gravity, it is evident that a vertical filament IK, of heavy particles of the fluid contained in $ABEF$, would exert at the point K a pressure which must diffuse itself equally throughout the whole extent of the fluid $BCDE$; that this pressure would be
404. exerted with equal force from below upward to repel the action of each of the other vertical filaments belonging to the several points of BE; hence the filament IK effects, by itself, an equilibrium with all the other filaments of the mass $ABEF$; therefore the mass $BCDE$ being destitute of gravity, there will result no other pressure on the bottom CD, than that arising from the filament IK, which transmitting itself equally to all the points of CD causes upon CD a pressure equal to that exerted at the

point K, repeated as many times as there are points in CD. Accordingly, if we suppose the heavy fluid contained in $ACDF$, divided into horizontal strata, the upper stratum would communicate to the bottom CD no other action than that which would be communicated by the filament $a\ b$; and the same being true of each stratum, the bottom CD will only receive the pressure that would arise from the sum of the filaments $a\ b$, $b\ c$, $c\ d$, &c.; and since this pressure would transmit itself equally to all the points of CD, it is equal to the area of CD multiplied by the sum of the pressures exerted upon some one point by the sum of the filaments $a\ b$, $b\ c$, $c\ d$, &c.; from all which we derive the following conclusions, namely; Fig.194.

(1.) *If the fluid* ACDF *be homogeneous, that is, composed of parts of the same nature, of the same gravity, &c., the pressure upon the bottom* CD *will be expressed by* CD $\times$ a g*; or, in other words, will be measured by the weight of the prism or cylinder which has* CD *for its base and* a g *for its altitude.*

(2.) *If the fluid is composed of strata of different densities, the pressure upon* CD *will be expressed by* CD *multiplied by the sum of the specific gravities of each stratum;* I say by the sum of the specific gravities, and not by the sum of the weights; for it is not on the quantity of the fluid contained in each stratum that the pressure depends but simply on the proper gravity of each filament.

It is important to observe that the above propositions hold true, whether the vessel grows larger toward the top, as in the present instance, or whether it is constructed from the bottom upward, as represented in figure 195. The pressure which the fluid contained in $ACDF$ exerts upon CD, is the same as if the cylinder $ECDG$ were filled with the fluid, the altitude being the same in both cases. This constitutes what is called the *hydrostatic paradox*, and is often expressed in the following words, namely; *any quantity of water or other fluid, however small, may be made to balance and support a quantity however large.* The principle is well illustrated by an instrument called the *hydrostatic bellows*; see figure 196, in which EF, CD represents two thick boards 16 or 18 inches in diameter, firmly connected together by pliable leather attached to the edges, which allows a

motion like that in common bellows. Instead of a valve, a pipe AB, about three feet in length, is inserted at B, either in the upper or lower part of the bellows. Now if water be poured into this pipe at A, it will descend into the bellows and gradually separate the pieces EF, CD, from each other by raising the latter; and if several weights, 200 pounds, for instance, be placed upon the upper board, the small quantity of water in the pipe AB will balance all this weight. More water being poured in, instead of filling the pipe and running over the top A, it will descend into the bellows, and slowly raise the weights; the distance between the surface of the water in the pipe and that in the bellows remaining the same as before. It is manifest from what has been said, that the upward pressure exerted upon each point of the interior surface of CD is sufficient to support a column of fluid of the same height with that contained in the tube AB, and consequently that the whole upward pressure, or weight sustained, is equal to the weight of a cylinder of water, whose base is the area of CD, and whose altitude is that of the column of water in AB above the surface of the water in the bellows or lower surface of CD. The area of the base, for instance, being a foot and a half, and the altitude three feet, the whole mass would be $4\frac{1}{2}$ cubic feet, and the weight sustained would be $4\frac{1}{2} \times 62\frac{1}{2}$* or $281\frac{1}{4}$ pounds, while the quantity contained in AB, depending on the size of the tube might weigh only one fourth of a pound or any less quantity.

It is obvious that instead of the gravity of the fluid in the tube AB, any other force might be employed, as the impulse of the breath, or that exerted by a stopper or *piston* moving in the tube AB. Thus, by means of a lever HI, a dense fluid, as water
Fig.197. for instance, might be forced through the pipe CC' against a large piston supposed to be accurately fitted to the cylinder FD, and connected with the rod or bar DE. A valve† being provided at F to prevent the return of the fluid, this action might be repeated; we should thus have an engine of almost unlimited

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*A cubic foot of water at the temperature of 50° weighs $1000^{\text{oz.}}$ avoirdupois or $62\frac{1}{2}^{\text{lb.}}$.

† See note on the construction of valves.

power contained at the same time within a small compass, and very simple in its construction. The diameter of the large piston being 12 inches, for example, and that of the small one worked by the lever H and moving in AB, only one fourth of an inch, the proportion of the two surfaces, or of the power employed to the force exerted at E, would be as $\frac{1}{16}$ to 144, or as 1 to 2304. Now it would be easy, by means of the lever HI, to apply to the small piston a force equal to 20 cwt. or one ton, in which case the piston working in FD would be moved with a force of 2304 tons. This instrument is called the *Hydrostatic* or *Bramah's Press*, Mr. Bramah, an Englishman, being the first person who made use of the hydrostatic principle here involved, as a substitute for the screw in the construction of presses.* This machine evidently belongs to the class of mechanical powers, and is essentially different in its nature from those heretofore described. The principle of virtual velocities, however, is equally applicable to this; since the greater the advantage gained in point of intensity, just so much is lost in respect to velocity. Suppose, for example, that the pipe AB, filled with fluid, is 2304 inches in length, the small piston, by moving through this whole extent, and thus forcing the entire contents of the pipe into the cylinder FD, would raise the large piston only one inch; so that while the pressure upon the small piston is to that upon the large one, in case of an equilibrium, as 1 to 2304, the spaces described in the same time, or the velocities of the two pistons, are as 2304 to 1, and the quantity of motion in each is the same.

411. Let there be two fluids $NHCBFL$, $EFLM$ of different densities, but each being homogeneous, considered by itself, and let them be made to act against each other at FL by means of the vessel in which they are contained. They can be in equilibrium only when the altitudes EF, IK, above the horizontal plane FL which separates them, are inversely as their specific gravities. Indeed, the fluid $LFBCGO$ being itself in equilibrium, it is necessary that $NHGO$ should be in equilibrium with $EFLM$; it follows, therefore, that the upward pressure exerted by the

Fig. 198.

407.

* The same property of fluids is sometimes employed very advantageously in a crane and in raising water from mines.

column $NHGO$ upon FL, should be equal to the downward pressure exerted upon FL by the column $EFLM$. Now the pressure of $NHGO$ upon FL is equal to the weight of a prism or cylinder of this fluid which has the surface FL for its base and IK for its altitude; moreover this weight is equal to the specific gravity multiplied by the bulk; accordingly, if we call the specific gravity S, we shall have for the expression of the weight $S \times IK \times FL$. For the same reason, if we call S', the specific gravity of the fluid $EFLM$, we shall have

$$S' \times EF \times FL,$$

as the expression for the absolute gravity of this fluid or the pressure which it exerts upon FL. Therefore

$$S \times IK \times FL = S' \times EF \times FL,$$

or

$$S \times IK = S' \times EF;$$

whence

$$S : S' :: EF : IK,$$

that is, the altitudes are inversely as the specific gravities. Thus if $LFBCHN$ were mercury, and $EFLM$ water; since mercury is 13,6 or nearly 14 times as heavy as water, the altitude IK would be one fourteenth part of EF, whatever be the figure of the vessel.

Fig.194. 412. From what has been said, it will be seen that the action of fluids is very different from that of solids. Properly speaking, it is only the part $ECDG$ which exerts its action upon the surface CD; and in figure 195, the surface CD is pressed by $ACDF$, as it would be by the weight of the fluid contained in the cylinder $ECDG$. If, on the contrary, the fluid $ACDF$ were suddenly to become a solid, by freezing, the bottom would support a pressure equal to the weight of the entire mass $ACDF$ in figure 194, and only equal to the weight of $ACDF$ in figure 195.

413. It is necessary here to distinguish between the force or pressure exerted on the bottom CD, and that which would be sustained by a person carrying the vessel. It is clear that if the bottom CD were movable, the only thing necessary to keep it in its place, would be an effort equal to the weight of the cylin-

der $ECDG$, but in order to transport the vessel, an effort would be required equal to the weight of the entire mass of water contained in the vessel. This will be demonstrated in a manner still more general after we have explained the method of estimating the pressure upon oblique plane surfaces and upon curved surfaces. Fig. 194.

414. Let $ACDF$ be the vertical section of a vessel terminated by surfaces either plane or curved, and inclined in any manner to the horizon. If we imagine an infinitely thin stratum $a\ b\ d\ c$, we can suppose it destitute of gravity, and pressed by the fluid above it. Now this pressure will be distributed equally to all points of the stratum, and will act perpendicularly and equally upon each of the points of the faces $a\ c$, $b\ d$. Accordingly, as this force is equal to that which a single filament IK would cause, the pressure exerted perpendicularly upon $b\ d$, will be expressed by $b\ d \times IK$; and it is evident that we should arrive at the same result, if instead of $b\ d$ being considered as a small straight line, we regard it as a small surface. We hence derive the general conclusion, that *the pressure exerted perpendicularly upon any infinitely small surface by a heavy homogeneous fluid, has for its expression this surface multiplied by its perpendicular distance from the level of the fluid.* Fig. 199, 200.

415. Hence the whole pressure exerted upon any plane surface, situated as we please, is equal to the sum of the infinitely small parts of this surface, multiplied each by its distance from the level of the fluid. If we represent these small parts by m, n, o, &c., and their distances respectively from the level of the fluid by AA', BB', CC', &c., according to article 76, we shall have

$$GG' \times (m + n + o + \&c.,)$$

$$= AA' \times m + BB' \times n + CC' \times 0 + \&c.,$$

that is, the sum of these products is equal to the whole surface multiplied by the distance of its centre of gravity from the same horizontal plane. Therefore *the pressure exerted by a heavy fluid against an oblique plane surface has for its measure the product of this surface into the distance of its centre of gravity from the line of level of the fluid.*

416. As the pressures exerted upon the several points of the same plane surface are perpendicular to the surface, and consequently parallel among themselves, the resultant or whole pressure must be parallel to the components. Now as we know how to find the resultant as well as that of each of the partial pressures, it will be easy to determine, as we have occasion, through what point the resultant passes; it evidently cannot pass through the centre of gravity, but must pass through some point lower
Fig. 201. down. It is only in the case where the surface is infinitely small that the whole pressure passes through the centre of gravity of this inclined surface.

417. In order to find the resultant of all these pressures both in a vertical and in a horizontal direction, any body, whatever its figure, may be considered as composed of an infinite number of strata, parallel among themselves, and the surface of the perimeter of each stratum may be represented by a series of trapezoids, of which the number is infinite, when the surface is a curve. So that in order to estimate the resultant of the pressure exerted by a fluid either upon the interior sides of a vessel, or upon the exterior surface of a solid immersed in it, we must determine the pressure exerted upon a trapezoid of an infinitely small altitude.

Accordingly, let us suppose a trapezoid $ABCD$, of which the
Fig. 202. two parallel sides are AB, CD, and the altitude infinitely small compared with these sides; and let there be applied at the centre of gravity G of the trapezoid perpendicularly to its plane, a force p equivalent to the product of the surface of this trapezoid into the distance of its centre of gravity from the horizontal plane XZ.

To determine the effect of this force, as well in a horizontal as in a vertical plane, suppose through the line CD a vertical plane $CDFE$, and through the line AB, considered as horizontal, a horizontal plane $AFEB$. Having drawn the vertical lines CE, DF, meeting this latter plane in E and F, we join BE, AF; and through the direction $G\,p$ of the force p, suppose a plane KIH cutting CD at right angles, HGK and HI being the intersections of this plane with the two planes $ABCD$, $FECD$, respectively. The plane KIH will be perpendicular to each of

the two planes $ABCD$, $FECD$, since CD is their common intersection; lastly, from the point K, where HK meets AB, let fall the perpendicular KI upon the plane $IECD$, this line must be perpendicular to HI. Geom. 355.

These steps being taken, I decompose the force p into two others, both being in the plane KIH produced, and of which one GL is horizontal or perpendicular to the plane $FECD$, and the other GM vertical. Calling these two forces q and r, and forming the parallelogram $GMNL$ upon the line GN, taken arbitrarily as the diagonal, we shall have

$$p : q : r :: GN : GL : GM,$$
$$:: GN : GL : LN.$$

But as the triangle GLN has its sides perpendicular respectively to those of the triangle KIH, these two triangles are similar, and we have Geom. 209.

$$GN : GL : LN :: HK : HI : IK;$$

and consequently

$$p : q : r :: HK : HI : IK.$$

Multiplying the three last terms by $\frac{AB + CD}{2} \times GG'$, which does not change the ratio, we obtain,

$$p : q : r$$
$$:: HK \times \frac{AB + CD}{2} \times GG' : HI$$
$$\times \frac{AB + CD}{2} \times GG' : IK \times \frac{AB + CD}{2} \times GG'.$$

We observe now, (1.) That $HK \times \frac{AB + CD}{2}$ is the surface of the trapezoid $ABCD$; (2.) That since CE, DF are parallel, as also CD, EF, CD is equal to FE; whence $IK \times \frac{AB + CD}{2}$ is equivalent to $IK \times \frac{AB + EF}{2}$, and consequently is the surface of the trapezoid $AFEB$; (3.) As we suppose the altitude of Geom. 178.

the trapezoid $ABCD$ infinitely small compared with the sides AB and CD, EF which is equal to CD, may be taken instead of AB and also instead of CD, so that $HI \times \frac{AB + CD}{2}$ reduces itself to $HI \times \frac{2\,EF}{2} = HI \times EF$, which is the surface of the rectangle $ECDF$; we have, therefore,

$$p : q : r :: ABCD \times GG' : ECDF \times GG' : AFEB \times GG'.$$

But we have supposed the force p expressed by $ABCD \times GG'$, hence the force q will be expressed by $ECDF \times GG'$, and the force r by $AFEB \times GG'$.

Since a triangle is simply a trapezoid, one of whose parallel sides is zero, the same results are applicable to a triangle.

Suppose now perpendiculars let fall from the angles A, D, C, B, upon the plane XZ. These perpendiculars may be considered as the edges of a truncated prism, the horizontal base being $AFEB$, and the inclined base $ABCD$. Now as AB, CD, are supposed to be infinitely near to each other, the bulk of this prism may be regarded as not differing from that of a prism of the same base, and whose altitude is GG'; but this last has for its expression $AFEB \times GG'$, which is precisely that just found for the vertical force r; therefore this force has also for its expression, the bulk of the truncated prism, whose inclined base is $ABCD$ and whose horizontal base is the projection of $ABCD$ upon the horizontal plane XZ.

418. Let any solid be divided into an infinite number of hor-
Fig. 203. izontal strata $ABDE\ a\ b\ d\ e$, and suppose that at the centre of gravity of the surface of each trapezoid of which the surface of the perimeter of this stratum is composed, forces are applied, represented each by the surface of the corresponding trapezoid multiplied by the distance of the centre of gravity from a horizontal plane XZ. These forces will represent the pressure ex-
415. erted by a heavy fluid upon the interior surface of the stratum $ABDE\ a\ b\ d\ e$ of a vessel in which this fluid is contained; they will also represent the pressure exerted by a similar fluid upon the exterior surface of a solid immersed in this fluid. Now we have seen that these forces being decomposed each into two

others, one vertical and the other horizontal, each vertical force will be represented by the truncated prism which has for its base the projection of the trapezoid upon the horizontal plane XZ, and for its inclined base, this trapezoid itself. Therefore the sum of the vertical forces, or the single vertical force that would result from them, will be represented by the sum of all the truncated prisms; and as the same reasoning is applicable to each horizontal stratum, we conclude, (1.) That *if a vessel* ABCDF, *of any figure whatever, be filled with a fluid to any line* AF, *there will result from all the pressures exerted by this fluid upon the several points of the vessel, no other vertical force than that which is represented by the bulk of this fluid, or rather by its weight.*

(2.) That *if a body, as* ACDBM, *for example, of which* AIBF Fig. 204. *is the greatest horizontal section, be immersed in a fluid to any depth whatever, the pressure exerted upon the superior part* AMB *being left out of consideration, the vertical effort of the fluid to raise the body, is equal to the weight of a volume of this fluid comprehended between the level* XZ, *the surface* AIBFC, *and the convex surface terminated by the perpendiculars let fall from the several points of the perimeter* AIBF *upon the plane* XZ.

If we next consider the pressure exerted upon the surface above the greatest horizontal section, it will be seen, by the same kind of reasoning, that there would result from the pressure of the fluid upon this surface in a vertical direction, a downward effort equal to the weight of a bulk of the fluid comprehended between this same surface, that of its projection $A'F'B'I'$, and that terminated by the perpendiculars let fall from the several points of the perimeter $AIBF$. Accordingly, if from the first vertical effort, we subtract the second, it will be seen that the body is urged vertically upward by an effort equal to the weight of a bulk of this fluid of which it occupies the place.

419. We hence derive the general conclusion, that *if a body be immersed in any fluid whatever, it loses a part of its weight equal to the weight of the fluid displaced, or equal to the weight of its own bulk of this fluid.*

420. There remain now two things to be inquired into; the first is, to determine through what point the vertical effort, result-

ing from the pressure of the fluid, passes; the second, to find what becomes of the horizontal forces.

(1.) It will be seen that the vertical effort must pass through the centre of gravity of the portion of fluid displaced. For, if we imagine this portion decomposed into an infinite number of vertical filaments, the vertical effort which the fluid exerts upon each filament, is expressed by the weight of a portion of fluid
272. equal to this filament; consequently, to find the distance of the resultant from any vertical plane, it is necessary to multiply the mass of each filament, considered as of the same nature with this fluid, by its distance from this plane, and to divide by the sum of the filaments. But this is precisely the course to be pursued, in
76. order to find the distance of the centre of gravity of the portion of fluid displaced; therefore the vertical effort of a fluid upon a body immersed in it, passes always through the centre of gravity of the portion of fluid displaced, which may be called the *centre of buoyancy.*

421. (2.) We proceed now to consider the horizontal forces above referred to. Representing always the solid stratum by figure 203, if through the sides *a b*, *b c*, &c., of the inferior section, we suppose vertical planes to pass terminating in the superior section; these planes will form the contour of a prism whose altitude is that of the stratum; and each face of this prism will
417. express by the extent of its surface the value of the horizontal force perpendicular to it. But, since all these faces are of the same altitude, their surfaces will be as their bases *a b*, *b c*, &c.,
Geom. 175. consequently the horizontal forces are to each other as the sides *a b*, *b c*, &c. Moreover, at whatever point of these faces they are applied, as these faces are of an altitude infinitely small, the horizontal forces may be considered as applied each in the horizontal plane *a b c d e f*, perpendicularly to the middle of the side which serves as a base to the corresponding face of the prism in question. I say to the middle, since it will readily be seen that the resultant of the pressures exerted upon the surface of any one of the trapezoids which form the surface of the stratum, must pass through some point of the line joining the middle points of the two parallel sides, and that, accordingly, the horizontal force obtained by decomposing this resultant, must meet the line joining

the middle points of the two opposite sides of the corresponding face of the prism. The problem, therefore, is reduced to finding what must take place in any polygon, when each of its sides is drawn or pushed by a force applied perpendicularly to its middle point, and represented as to its value by this side. We shall see that they will mutually destroy each other.

Let us, in the first place, consider only the two forces p and q, Fig. 205.
applied perpendicularly to the middle points of the sides AB, AC, of the triangle ABC, these forces being represented as to their values by these sides respectively. It is clear that their resultant would pass through their common point of meeting F, which, in the present case, is the centre of a circle in whose cir- Geom.
cumference the points A, B, C, are situated. We say, moreover, 106.
that this resultant will pass through the middle point of BC, to which it will consequently be perpendicular, and that it will be represented in magnitude by BC. For, if we decompose the force p into two others, one $D\,e$ parallel, and the other $D\,h$ perpendicular, to BC, by forming the parallelogram $D\,e\,g\,h$ we shall have, by calling these two forces e and h respectively,

$$p : e : h :: Dg : De : Dh :: Dg : De : ge.$$

Now, by letting fall the perpendicular AO, the triangle $g\,e\,D$ is similar to the triangle AOB, since their sides are respectively perpendicular. Accordingly,

$$Dg : De : ge :: AB : AO : BO,$$

whence

$$p : e : h :: AB : AO : BO.$$

But, by supposition, the value of the force p is represented by AB; therefore that of e is represented by AO, and that of h by BO.

If we decompose in like manner the force q into two others, the one $I\,m$ parallel, and the other $I\,k$ perpendicular, to BC, it may be shown as above, that m is represented by AO, and k by CO. The two forces m and e are therefore equal, since they are represented by the same line AO. Moreover they act in opposite directions, and according to the same line DI parallel

to BC, since D, I, are the middle points of AB, AC, respectively. Consequently, they will destroy each other. The resultant then must be the same as that of the two forces h and k; and as these are parallel, being each perpendicular to CB, their resultant must be equal to their sum, and perpendicular to BC; that is, (1) It will be represented by $BO + OC$ or by BC; and (2.) being perpendicular to BC, and passing, as we have just seen, through the centre F of the circle circumscribed about ABC, it passes through the middle point BC.

Fig. 206. This being premised, the resultant ϱ of the two forces p, q', will be perpendicular to the middle of BE, and be represented by BE. For the same reason, the resultant ϱ' of the two forces ϱ, p', and consequently of the three forces p, q', p', will be perpendicular to the middle of BD and be represented by BD. Lastly, the resultant ϱ'' of the forces ϱ', q, and consequently of the forces p, q', p', q, will be perpendicular to the middle of DC and be represented by DC; it will accordingly be equal and directly opposite to the force r; therefore all these forces will destroy each other. The same reasoning will evidently be applicable, whatever the number and magnitude of the sides. Hence we derive the general conclusion, that *the efforts which result in a horizontal direction from the pressure of a heavy fluid exerted perpendicularly upon the surface of a body immersed in it, mutually destroy each other.*

422. The pressures upon any given surface being considered by themselves, the distance at which the resultant, or the *centre of pressure* passes, is readily found. The forces exerted upon the several points being as the distances respectively of these points from the surface of the fluid, they are as the forces that would arise from the motion of this surface about the intersection of its plane with the surface of the fluid as an axis. But we have seen that the distance of the resultant in this case is that of the centre of per-
357. cussion or oscillation. Hence the distance of the centre of pressure of any given surface from the surface of the fluid, is the same as that of the centre of percussion. Accordingly, the centre of pressure
Fig. 176. on the side of a perpendicular prismatic vessel, is one third of the
365. height from the bottom.

Moreover, since the magnitude of this pressure is equal to the surface multiplied by the distance of the centre of gravity, the pres- 415. sure on the upright side of a vessel of the form of a parallelopiped is to the pressure on the bottom, as half the area of the side Fig.176. to the area of the bottom, or as half the height of the side to the length of the bottom, reckoned in the direction of a perpendicular to this side. When the parallelopiped is a cube, the pressure on the side is half that on the bottom, and the pressure upon the four sides together, double that upon the bottom.

Since the pressure of any fluid is proportional to the depth below the surface, the strain upon the sides of a sluice, and the banks of a canal, must increase uniformly from the top to the bottom; and, when this force is to be resisted by earth or masonry, since the strength of the materials may be estimated in a certain proportion to the weight, commonly as a third or fourth part, if 208 Fig *AB* be the height of the bank or dike, and a third of its density Fig.20 be to that of water, as *AB* to *BC*, by joining *AC*, this line will represent the proper slope. If the bank be composed of stone or brick, the base *BC* must be at least equal to the altitude *AB*; if it be of earth, *BC* should exceed the height *AB* by one half.

Let *AC*, *BC*, represent the floodgates of a canal-lock which Fig.209. are opened by means of the extended arms *AF*, *BF*. When we shut the gate, *AC* is pressed at right angles by the water with a force proportional to *AC*, and which, from the principle of the moment of inertia, must exert a perpendicular effort at the end *C*, as $\overline{AC}^2$. The strain thence produced in the direction *AB* 356. will be opposed by an equal and opposite effort from the gate *BC*. These two forces constitute the power which closes the gates. If the angle *ACB* be very acute, the gates would close feebly; if on the other hand *ACB* be too obtuse, the gates would occasion a great strain upon the sides of the sluice. This strain in the direction *CA* is as $\overline{AC}^2$, other things being the same; it is also as the width of the canal or as *AD*, and for the reason above given, inversely as *CD*; that is, the force in question is as

$$\frac{\overline{AC}^2 \times AD}{CD},$$

or, the width of the canal being constant, as $\frac{\overline{AC}^2}{CD}$. We have, therefore, only to find when this is a minimum. It is evidently equal to the diameter of the circle circumscribing the triangle
Geom. 230. ACB; and since the least circumscribing circle is that whose centre is D, we infer that the angle ACB of the meeting of the gates should be a right angle.

423. Water is sometimes conveyed in pipes, which, according to what is above said, must sustain a pressure in proportion to the height of the source. There is moreover a force exerted
Fig. 210. upon the interior of the pipe depending upon its diameter. ADE being a transverse section of the pipe, let A be a particle of the fluid pressed by two contiguous particles B, C, and kept in equilibrium by the resistance or tenacity of the substance of
48. the pipe. These three forces will be represented by the three sides of a triangle formed upon their directions, or by the three sides of the similar triangle OBC, formed by lines drawn perpendicularly to the directions of the forces. Now if we suppose BC to become indefinitely small, or that the forces exerted by B, C, are directly opposed to each other, each will be represented by the radius of the section. Thus, the height of the source being the same, and the substance of the pipe the same, its thickness ought to be in proportion to the radius or diameter of the bore.

We may apply the same conclusion to cylindrical vessels generally destined to hold fluids. Large casks are required to be made stronger than small ones, in the compound ratio of their diameter and height. The same precaution is to be observed, moreover, with regard to steam-pipes and steam-boilers of different dimensions, for the above reasoning will hold true equally in the case of elastic fluids, as in that of liquids.

Solids immersed in Fluids.

424. Since the efforts which a fluid makes in a horizontal direction mutually destroy each other, in order to preserve a

body in any given position in a fluid, it is only necessary to destroy the vertical part of the pressure; and to effect this, two things are required, namely, (1.) A downward effort equal to that of the upward pressure of the fluid; and (2.) A coincidence of these efforts in the same vertical line. Now the vertical upward pressure or buoyancy of the fluid is equal to the weight of the portion of fluid displaced; hence, *if the portion of fluid displaced weighs more than the immersed body, the body will float, and it will elevate itself until the portion of fluid answering to the part immersed, shall weigh just as much as the entire body.*

Accordingly, if, when a body floats, we add to it a certain weight, or take a certain weight from it, it will sink or rise until the increase or diminution of the fluid displaced shall become equal to this new weight. If the weight added or subtracted be small compared with that of the body itself, the quantity IK, by which the section AB is depressed or elevated, will be so Fig. 207.
much the less, according to the smallness of this new weight compared with the extent of the section AB. When, therefore, this new weight is inconsiderable, and the section AB is large, AB and $A'B'$ may be considered as equal, and the difference in the bulk of fluid displaced, occasioned by the supposed change of weight, may be estimated by the surface AB multiplied by IK, that is, by $AB \times IK$. Therefore if w represent the weight of a cubic foot of the fluid,* $w \times AB \times IK$ will express the weight of the bulk in question, the surface AB, and the altitude IK being estimated in feet. Thus, if w' be the weight added or subtracted, we shall have

$$w \times AB \times IK = w',$$

from which we deduce

$$IK = \frac{w'}{w \times AB};$$

that is, in the case of a vessel, for example, in order *to find how much a certain addition to the cargo will sink the vessel,* we divide

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* This in the case of fresh water is at a mean very nearly $62\frac{1}{2}^{\text{lb.}}$, and in that of sea-water $64^{\text{lb.}}$.

the value of this addition w', by the surface of the section made at the water's edge (estimated in square feet), multiplied by the weight of a cubic foot of water.

When a vessel is sheathed, the bulk of the hull is augmented by a quantity, the method of calculating which has
Cal. 122. been made known. The effect, therefore, is the same, as if the cargo were diminished by the following quantity, namely, the weight of a mass of water equal to the augmentation of the hull, minus the weight of the materials which compose the sheathing. Accordingly, it may easily be determined how much less or more the vessel would sink.

425. The weight of a body remaining the same, its bulk may be enlarged at pleasure, by forming it so as to inclose a space. There is no substance therefore so heavy that it may not be made to float.

426. Since, when the weight of a body is diminished without changing its bulk, it must elevate itself with an effort which can be counterbalanced only by a weight equal to that which has been taken from it, it will be seen that this upward pressure of water may be advantageously employed in raising large masses; in drawing up vessels, for example, from the bottom of bays and rivers, by attaching them to other vessels, floating above, and deeply laden with stones or water, afterward to be thrown overboard.

Specific Gravities.

427. In general, if S be the specific gravity of a floating body, or, which is the same thing, the weight in ounces of a cubic foot of this body,* the matter being supposed to be uniformly distributed, b its bulk, S' the specific gravity of the fluid,

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* Water is the unit to which we refer all substances, except the gases, in estimating their specific gravities. It is immaterial what bulk be used for this purpose, or whether any particular bulk, provided the two in question be equal. A cubic foot of any

and b' the bulk of the part immersed, the weight of this body will be $S\,b$; and that of the fluid displaced will be $S'\,b'$; whence we have

$$S\,b = S'\,b',$$

an equation which gives

$$b' = \frac{S\,b}{S'};$$

from which it will be seen, that the weight $S\,b$ of the body remaining the same, the part immersed will always be so much the less, according as the specific gravity of the fluid is greater.

Moreover, this same equation is equivalent to the proportion,

$$b : b' :: S' : S;$$

that is, the bulk of the body is to that of the part immersed, inversely as the specific gravity of the body to that of the fluid.

428. If the immersed body weigh more than an equal bulk of the fluid, it must sink; and it can be retained only by a force equal to the excess of its weight above that of an equal bulk of the fluid. Now if we represent the specific gravity of the fluid,

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substance compared in point of weight with the same bulk of water would evidently give the same result as any other measure; and correct results once obtained, we can substitute such other bulks as we choose. The cubic foot has this particular advantage, that, if the measures be taken at the temperature of 50° of Fahrenheit, the same numbers which express the specific gravities, the decimal point being removed three places, or the whole being multiplied by 1000, give the absolute weight in avoirdupois ounces; since a cubic foot of water at this temperature weighs 1000 ounces. It is usual, however, in determining specific gravities, to refer to the temperature of 60°, at which a cubic foot of water weighs $62{,}353^{\text{lb.}}$, or a little less than 1000 ounces. In the more accurate experiments, however, upon this subject, the absolute weight is first ascertained, and the cubic inch is taken as the measure, this bulk of distilled water at the temperature of 60°, being estimated at 252,525 grains. In France the temperature preferred is that of the maximum density of water, or 39°,39. The atmosphere is employed as the unit in estimating the specific gravities in gases.

and that of the body immersed in it, by S', S, respectively, and the bulk of the body by b, we shall have, $S\,b - S'\,b$ for the excess of the weight of the body above that of an equal bulk of the fluid. Hence, if we suppose that this body is retained by means of a thread, attached to the beam of a balance, and that w is the weight with which it is kept in equilibrium; we shall have

$$w = S\,b - S'\,b,$$

from which we obtain,

$$\frac{S}{S'} = \frac{S\,b}{S\,b - w}.$$

Now $S\,b$ is the absolute weight of the body, and w its weight in the fluid; *knowing, therefore, the absolute weight of a body, and its weight when immersed in a fluid, we easily determine the ratio of their specific gravities, by dividing by the absolute weight of the body the difference of these two weights.* If, for example, the absolute weight of a body be 6 ounces, and its weight when immersed in water 5 ounces, we divide the absolute weight 6 by the difference, and the quotient $\frac{6}{1}$ shows that the specific gravity of the body is 6 or is to that of the fluid as 6 is to 1.

In other words, since the loss of weight sustained by a body on being immersed in a fluid is equal to the weight of its own bulk of that fluid, by proceeding as above, we shall have the weight of equal bulks of the substances in question, and consequently their relative weights or specific gravities.

If the solid substance whose specific gravity is sought be lighter than water with which it is to be compared, it may be made to sink by attaching to it a heavier body, whose *water-weight* is known; then, by subtracting this from the water-weight of the compound body, we shall have that of the body in question.

When the substance to be weighed is soluble in water, as common salt, for instance, it may be covered with a thin coat of melted wax, which is very nearly, and may be made exactly of the same specific gravity with water. The body will thus be protected, and the loss of weight, on being immersed, will be the same as if water occupied the place of the covering.

Another method is to determine the specific gravity of the body with reference to some liquid, as alcohol or oil, in which no solution takes place, and whose specific gravity, compared with water, is known. We have then simply to use the proportion, as the specific gravity of water, is to that of the fluid used so is the result above found to the result sought.

429. If the same body be immersed in another fluid, whose specific gravity is denoted by S'', and w' represent the weight necessary to counterbalance it; as in the former case we had $w = S\,b - S'\,b$, we shall have, in like manner, $w' = S\,b - S''\,b$. Now these two equations give

$$S'\,b = S\,b - w, \quad \text{and} \quad S''\,b = S\,b - w';$$

whence, dividing this last by the preceding, we obtain

$$\frac{S''}{S'} = \frac{S\,b - w'}{S\,b - w}.$$

Knowing, therefore, the absolute weight $S\,b$ of a body, and its weight w' in a fluid, and its weight w in any other fluid, we easily determine the ratio $\frac{S''}{S'}$ of the specific gravities of these two fluids.

By taking a solid glass ball and grinding it to such a size that it shall lose just a thousand grains, for instance, when weighed in distilled water, at the assumed temperature, then by observing the loss of weight l in grains, sustained on being weighed in any other fluid, we shall have

$$1000 : l :: 1 : S = \frac{l}{1000}.$$

Thus the specific gravity of the fluid in question is obtained by dividing l by 1000, or removing the decimal point three places to the left. If a larger number of decimal places were required, we should employ a ball weighing 10000 or 100000 grains, and divide accordingly.

430. The precious metals being heavier than those of less value, when they are debased, it must be by means of some substance of less specific gravity, and the compound will consequently be lighter than the metal it is intended to represent.

Spirits, on the other hand, are lighter than the liquids with which they are adulterated, so that, in each case, the specific gravity becomes the test of purity. We have, hence, a curious and important problem, namely, knowing the specific gravity of the two ingredients that compose a compound, and the specific gravity of the compound, to find the proportion of the ingredients. This proportion may be estimated by weight, as is usually the case with respect to metals, or by measure, which is the common method where liquids are concerned. The weight of the ingredients being sought, we put x for that of the denser, and y for that of the rarer or lighter, c representing that of the compound, and S, S', S'', denoting their specific gravities respectively. We have $x + y = c$; also, on the supposition that the bulk of the compound is equal to that of the constituent parts, the specific gravity would be the same whether the compound were considered as one entire mass, or as composed of two distinct parts, and the weight divided by the specific gravity in the one case would be equal to the weight divided by the specific gravity in the other, that is,

$$\frac{x}{S} + \frac{y}{S'} = \frac{c}{S''}.$$

To find x, we substitute in this equation the value of y deduced from the first, namely, $y = c - x$, which gives

$$\frac{x}{S} + \frac{c - x}{S'} = \frac{c}{S''},$$

or

$$\frac{x\,(S' - S)}{SS'} = \frac{c}{S''} - \frac{S\,c}{S\,S'} = \frac{(S' - S'')}{S'\,S''}\,c,$$

whence

$$x = \frac{SS'}{(S' - S)} \times \frac{(S' - S'')}{S'\,S''}\,c = \frac{(S' - S'')\,S}{(S' - S)\,S''}\,c = \frac{(S'' - S')\,S}{(S - S')\,S''}\,c.$$

In like manner, we have

$$y = \frac{(S - S'')\,S'}{(S - S')\,S''}\,c.$$

Suppose a compound of gold and silver to have a specific gravity equal to 14, that of pure gold being 19,3, and that of pure silver 10,5; we have

$$S = 19{,}3, \quad S' = 10{,}5, \quad S'' = 14;$$

and consequently,

$$\begin{aligned} S'' - S' &= 3{,}5, \\ S - S' &= 8{,}8, \\ S - S'' &= 5{,}3; \end{aligned}$$

whence, by the formula, the proportional weight of gold will be

$$\frac{3{,}5 \times 19{,}3}{8{,}8 \times 14} = \frac{67{,}55}{123{,}2} = 0{,}55.$$

The formula for the value of y gives for the proportional weight of silver

$$\frac{5{,}3 \times 10{,}5}{8{,}8 \times 14} = \frac{55{,}65}{123{,}2} = 0{,}45,$$

whatever c may be in each case.

431. Where the proportion of the ingredients by bulk or measure, instead of by weight, is sought, as in the case of liquids, the process is shorter. Calling b, b', the bulks of the ingredient corresponding to the specific gravities S, S', as the weight is equal to the bulk multiplied by the specific gravity, and the weight of the compound is equal to the sum of the weights of the ingredients, we have

$$S'' \times (b + b') \quad \text{or} \quad S''\,b + S''\,b' = S\,b + S'\,b',$$

whence

$$S''\,b - S\,b = S'\,b' - S''\,b',$$

or

$$b\,(S'' - S) = b'\,(S' - S'');$$

that is,

$$b : b' :: S' - S'' : S'' - S.$$

Suppose a certain spirituous liquor to have a specific gravity equal to 0,93, that of highly rectified alcohol or pure spirit being

0,83 and water 1; we have in this case

$$S = 0{,}83, \quad S' = 1, \quad \text{and} \quad S'' = 0{,}93,$$

which gives,

$$S' - S'' = 0{,}07, \quad S'' - S = 0{,}1,$$

and by substitution,

$$b : b' :: 0{,}07 : 0{,}10;$$

that is, the proportion by measure of pure spirit to that of water, is as 7 to 10. In what is called *proof* spirit, it is required that the constituent parts of water and spirit should be equal, which gives a specific gravity of 0,925, and the degree above or below proof is usually denoted by the number of gallons of water to be added to or taken from 100 gallons of the liquor in question, to bring it to the required standard or proof.

But more expeditious methods have been devised for determining the proportion of alcohol in spirits. It is evident that if a great variety of mixtures of known proportions were prepared, and hollow glass balls or *beads* were so adapted to each in respect to specific gravity, as just to remain suspended in any part of the fluid; by marking the known proportion of alcohol and spirits in each case on the respective balls, similar unknown mixtures might be readily examined as to their proportion of alcohol. It would only be necessary to make trial of different balls until one was found which would tend neither to rise nor sink when immersed in the fluid. But instead of a large number of such balls, a single one with a long graduated stem and movable weights, as represented in figure 211, might be employed to the same purpose. The ball itself should be of such a specific gravity as just to sink to the commencement of the graduation on the stem when suspended in alcohol, and the weights to be attached should be such as to cause the ball and some part of the stem to sink in any mixture of less specific gravity than pure water. Then the weight together with the divisions of the stem, which serve to subdivide the difference between two weights, would indicate, like the separate balls, the specific gravity of the

liquid and consequently its proportion of alcohol. An instrument so constructed, is called a *hydrometer.**

432. In making use of the hydrometer, and in experiments generally upon specific gravities, there are several particulars to be taken into consideration.

(1.) Since all bodies expand with heat and contract with cold, it will be perceived that the specific gravity of bodies is modified by temperature. Thus bodies are specifically heavier in winter than in summer, and the same spirit would indicate different proportions of alcohol at different seasons, regard not being had to this circumstance. Accordingly a thermometer is a necessary appendage to a hydrometer, and the correction for temperature is applied by means of a movable scale, containing the degrees of the thermometer, sliding upon another scale on which are placed the numbers of the several weights, including the stem.

(2.) When two substances are mixed together, there is often a mutual penetration of parts, whereby the specific gravity is increased, and there would seem, by the foregoing methods, to be a greater proportion of the heavier ingredient than actually exists. Thus a pint of water and a pint of alcohol do not make a quart of liquid. The defect is sometimes a 40th part. On the other hand, the bulk in certain cases is augmented by compounding. A cubic inch of tin mixed with a cubic inch of lead will make a compound exceeding two cubic inches. No accurate allowance can be made for such changes; consequently the methods above given, become in a degree defective, and the results to a certain extent uncertain.

(3.) The number of ingredients may exceed two, or the nature of one or more of the ingredients may be unknown. In all such cases, the problem is in its nature indeterminate.

433. Questions sometimes occur, especially in chemical researches, the reverse of those we have been considering, namely,

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* There is a great variety of instruments of this kind, all depending on the same general principles. They are known also by a variety of names, as *areometer*, *gravimeter*, *alcoholometer*, *pese-liqueur*, *essay-instrument*, &c.

to find the specific gravity of a compound, by means of that of each of the ingredients; and the rule which has generally been given, is to take the arithmetical mean of the specific gravities of the ingredients.

It will be observed that, the greater the weight of a body, other things being the same, the greater the specific gravity; also that the less the bulk, other things being the same, the greater the specific gravity; that is, the specific gravity of one body is to that of another, as the weight of the first divided by its bulk, is to the weight of the second divided by its bulk, and hence the mean specific gravity of the two will be found by dividing the sum of the weights by the sum of the bulks. Thus,

$$S = \frac{w}{b}, \quad \text{and} \quad S' = \frac{w'}{b'},$$

from which we obtain,

$$b = \frac{w}{S}, \quad \text{and} \quad b' = \frac{w'}{S'};$$

also,

$$b + b' = \frac{w}{S} + \frac{w'}{S'} = \frac{w\,S' + w'\,S}{SS'}.$$

Hence, calling M the mean specific gravity sought, from the equation $M = \frac{w + w'}{b + b'}$, above shown to be true, by general reasoning, we obtain, by substitution,

$$M = \frac{w + w'}{\frac{w\,S' + w'\,S}{SS'}} = \frac{(w + w')\,SS'}{w\,S' + w'\,S}.$$

Let gold, for example, of a specific gravity 19,36, be alloyed in equal weights with copper of a specific gravity 8,87, we shall have

$$M = \frac{(1 + 1)\,19{,}36 \cdot 8{,}87}{8{,}87 + 19{,}36} = \frac{2 \times 171{,}7232}{28{,}23} = 12{,}16;$$

whereas the arithmetical mean is

$$\frac{19{,}36 + 8{,}87}{2} = 14{,}11.$$

434. It will be recollected that the specific gravity of a solid body is determined by dividing its absolute weight by the loss sustained on its being weighed in pure water, and the specific gravity of a fluid, whether liquid or gas, by comparing the loss of weight sustained by the same solid when weighed in pure water with that sustained on its being weighed in the fluid in question. According to this method, therefore, the absolute weight of a body is necessary in both cases. But it is difficult to obtain this unaffected by the fluid of the atmosphere in which we are immersed. The usual operation of weighing, except where the weight itself and the thing weighed happen to be equal in bulk, must, from what has been said, be more or less incorrect. But the specific gravity of the atmosphere being ascertained, on the supposition that it is always the same, or such as to admit of its changes of density being determined at all times, allowance may be made for its effect on the weight of bodies, more especially as it is an exceedingly light fluid, and scarcely requires to be noticed, except in very nice experiments, or where the bulks of bodies are very considerable. The best way of determining the specific gravity of the atmosphere, and of gases generally, is to weigh directly a vessel of known dimensions, when empty, and again when filled with the fluid in question.* It is thus found that a vessel of three hundred cubic inches, for example, weighs 92,4 grains more when filled with air in its ordinary state, than

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* The vessel should be of considerable size, that is, sufficient to contain at least three or four hundred cubic inches. It might be of a globular shape, as represented in figure 212, with a narrow neck and nicely fitted stop-cock *C*. Its capacity would be best ascertained by filling it accurately with mercury, and then pouring the liquid into a prismatic vessel, which might be easily measured. The air might be expelled also in the same way, by connecting with the neck *AB* a tube of about three feet in length, and suffering the mercury to discharge itself through this tube, held upright with its lower end immersed in the same liquid. When the mercury had left the ball, upon turning the stop-cock we should effectually exclude the air; but there would remain a small portion of the vapor of mercury. Ordinarily the air is exhausted by means of an air-pump, and, although the air cannot thus be wholly withdrawn, the small proportion which is left may be measured and allowance made accordingly, as will be shown hereafter.

it does when empty. This divided by 300, or $\frac{92,4}{300}$, gives for the absolute weight of a cubic inch of air 0,308 parts of a grain. By dividing this by 252,525, the weight in grains of a cubic inch of water, gives 0,00122 or $\frac{1}{820}$ nearly for the specific gravity of common air, at the surface of the earth in its mean state of density and moisture, the temperature being that to which specific gravities are generally referred by English philosophers, namely, 60° of Fahrenheit. Hence bodies weighed in air lose at a mean $\frac{1}{820}$ part of the weight lost on being weighed in water. Accordingly, if we weigh a body in air and increase this weight by $\frac{1}{820}$ of the difference between the air and water weight, we shall have the absolute weight very nearly; that is, if w be the absolute weight, w' the air weight, and w'' the water weight, we shall have $w = w' + \frac{1}{820}(w' - w'')$ very nearly.*

If it were proposed to find how much a solid ball must weigh in the air in order that its absolute weight may be 1000 grains, from the equation

$$1000 = w' + \tfrac{1}{820}(w' - w''),$$

we should have

$$w' = 1000 - \tfrac{1}{820}(w' - w''),$$

or,

$w' - w''$ being 328, for example, $w' = 1000 - 0,4 = 999,6$.

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* To be strictly correct, the formula should be

$$w' + \tfrac{1}{820}(w - w''),$$

but the object of this formula is to find w, and when we have obtained it nearly, we may substitute this value, and thus approximate the true value of w to any degree of exactness. But generally speaking, the correction derived from the second approximation is very small. Thus, in the example that follows above, $\frac{1}{820}(w' - w'')$ is 0,4 grains, and the second approximation would give only $\frac{1}{820}$ of 0,4 grains or 0,0005 of a grain. Where the results are intended to be very accurate, the coefficient $\frac{1}{820}$ should be corrected for the particular state of the atmosphere at the time of the experiment, the method of doing which depends upon instruments to be described hereafter.

435. According to what is above laid down, the specific gravity of a body multiplied by its bulk gives its weight. Now the density multiplied by the bulk gives what is called the mass of the body, which is proportional to its weight. Hence the specific gravity multiplied by the bulk, is proportional to the density multiplied by the bulk; therefore *the specific gravities of bodies are proportional to their densities*. Thus S, S', being the specific gravities, w, w', the weights, b, b', the bulks, Δ, Δ', the densities, and m, m', the masses of two bodies, 427. 30. 272.

$$S\,b = w, \quad \text{and} \quad S'\,b' = w';$$

also,

$$\Delta\,b = m : \Delta'\,b' = m' :: w : w';$$

whence

$$\Delta\,b : \Delta'\,b' :: S\,b : S'\,b',$$

or,

$$\Delta : \Delta' :: S : S'.$$

By employing the same unit in both cases, as water, for example, at the same temperature, the specific gravities would be equal to the densities.

If fluids of various densities, and not disposed to unite by any chemical affinity, be poured into a vessel, they will arrange themselves in horizontal strata, according to their respective densities, the heavier always occupying the lower place. This stratified arrangement of the several fluids will succeed, even though a mutual attraction should subsist, provided only that its operation be feeble and slow. Thus, a body of quicksilver may be at the bottom of a glass vessel, above it a layer of concentrated sulphuric acid, next this a layer of pure water, and then another layer of alcohol. The sulphuric acid would scarcely act at all upon the mercury, and a considerable time would elapse before the water sensibly penetrated the acid, or the alcohol the water. Bodies of different densities might remain suspended in these strata. Thus, while a ball of platina would lie at the bottom of the quicksilver, an iron ball would float on its surface; but a ball of brick would be lifted up to the acid and a ball of beech would swim in the water, and another of cork might rest on the top of the alcohol.

Specific Gravities of the more remarkable Substances.

Platinum, purified, . . .	19,50	Plumbago, ...	1,86
——— hammered, .	20,33	Alum, ...	1,72
——— laminated, . .	22,07	Asphaltum, ...	1,40
——— drawn into wire,	21,04	Jet, ...	1,24
Gold, pure and cast, . .	19,26	Coal, from . . .	1,24 to 1,30
—— hammered, . . .	19,36	Sulphuric acid, ...	1,84
Mercury, ...	13,57	Nitric acid, ...	1,22
Lead, cast, ...	11,35	Muriatic acid, ...	1,19
Silver, pure and cast, . .	10,47	Equal parts by weight of	
—— hammered, . .	10,51	water and alcohol, . .	0,93
Bismuth, cast, ...	9,82	Ice, ...	0,92
Copper, cast, ...	8,79	Strong alcohol, ...	0,82
——— wire, ...	8,89	Sulphuric æther, ...	0,74
Brass, cast, ...	8,40	Naphtha, ...	0,71
—— wire, ...	8,54	Sea water, ...	1,03
Cobalt and nickel, cast, .	7,81	Oil of sassafras, ...	1,09
Iron, cast, ...	7,21	Linseed oil, ...	0,94
Iron, malleable, ...	7,79	Olive oil, ...	0,91
Steel, soft, ...	7,83	White sugar, ...	1,61
—— hammered, . . .	7,84	Gum Arabic and honey .	1,45
Tin, cast, ...	7,30	Pitch, ...	1,15
Zinc, cast, ...	7,20	Isinglass, ...	1,11
Antimony, cast, . . .	4,95	Yellow amber, ...	1,08
Molybdænum, ...	4,74	Hen's egg, fresh laid, . .	1,05
Sulphate of barytes, . .	4,43	Human blood, ...	1,03
Zircon of Ceylon, . . .	4,41	Camphor, ...	0,99
Oriental ruby, ...	4,28	White wax, ...	0,97
Brazilian ruby, ...	4,53	Tallow, ...	0,94
Bohemian garnet, . . .	4,19	Pearl, ...	2,75
Oriental topaz, ...	4,01	Sheep's bone, ...	2,22
Diamond, ...	3,50	Ivory, ...	1,92
Crude manganese, . . .	3,53	Ox's horn, ...	1,84
Flint Glass, ...	2,89	Lignum vitæ, ...	1,33
Glass of St. Gobin, . . .	2,49	Ebony, ...	1,18
Fluor Spar, ...	3,18	Mahogany, ...	1,06
Parian marble, ...	2,34	Dry oak, ...	0,93
Peruvian emerald, . . .	2,78	Beech, ...	0,85
Jasper, ...	2,70	Ash, ...	0,84
Carbonate of lime, . . .	2,71	Elm, ...	from 0,80 to 0,60
Rock crystal, ...	2,65	Fir, ...	from 0,57 to 0,60
Flint, ...	2,59	Poplar, ...	0,38
Sulphate of lime, . . .	2,32	Cork, ...	0,24
Sulphate of soda, . . .	2,20	Chlorine, ...	0,00302
Common salt, ...	2,13	Carbonic acid gas, . .	0,00164
Native sulphur, ...	2,03	Oxygen gas, . . .	0,00134
Nitre, ...	2,00	Atmospheric air, . .	0,00122
Alabaster, ...	1,87	Azotic gas, . . .	0,00098
Phosphorus, ...	1,77	Hydrogen gas, . . .	0,00008

It will be seen by the foregoing table, that there are gases much lighter than the atmosphere; accordingly, if a large quantity of one of these fluids were confined by a thin covering, as oiled silk, it would rise in the atmosphere as a cork does in water until it reached a region of the atmosphere of the same specific gravity with itself. It is upon this principle that balloons are constructed. They were at first filled with air rarefied by heat, and a fire was supported in a car placed under the balloon for this purpose. Hydrogen was afterward discovered, and proved to have a specific gravity only $\frac{1}{15}$ of that of common atmospheric air. This gas is the lightest of known fluids, and will consequently require a less bulk for a given buoyancy than any other. It is universally employed in the construction of balloons.

Spirit Level.

436. This instrument consists of a tube AB, nearly filled with alcohol or ether, the remaining space CD being occupied by a bubble of air. Being attached to the vertical or horizontal circle of a theodolite or other instrument, it serves to determine the horizontal position of a line or plane, since if either end of the tube is higher than the other, the bubble of air, from its specific lightness, will indicate it by tending to the higher part. It is usual, where great exactness is required, to make the tube slightly curved, the curvature being circular; the bubble will then move more readily, settle itself with more certainty, and describe equal spaces by equal changes of inclination. A good spirit level will exhibit a movement of more than half an inch for each minute of inclination, and the bubble will sensibly alter its position by a change of five seconds in the inclination. In such a tube, the radius of curvature will be about 150 feet. But levels have been rendered still more delicate. Lalande speaks of one, filled with ether, the bubble of which passed over fourteen inches by equal spaces of one tenth of an inch for every second. The radius of curvature was consequently 1719 feet, or nearly one third of a mile.* Fig. 213.

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* The number of seconds in a circle is

$$360 \times 60 \times 60 = 1296000.$$

Of the Equilibrium of Floating Bodies.

437. In order that a heavy body may be in equilibrium on the surface of a tranquil fluid, it is necessary that its weight should be less than that of an equal bulk of the fluid. There is an exception to this rule, however, in the case of very small bodies, the particles of which are of a nature not to exert any attraction upon the particles of the fluid, or whose attraction is much less than that of the particles of fluid for each other. In this case the fluid is depressed below its level, forming a little hollow about the floating body, which may be regarded as making a part of the bulk of the body; and on account of this augmentation, it is evident that the body may float, although its specific gravity should be greater than that of the fluid. It is on this account that a needle smeared with tallow may be made to float on water. Small globules of mercury also are supported in a similar manner. To render this effect of no avail, we have only to suppose the floating bodies so large, that the void space which may be formed about them (and which is always very small) may be neglected when taken in connection with a bulk so much greater.

438. The weight of a body being smaller than that of an equal bulk of the fluid, the body will sink till the weight of the fluid displaced becomes equal to that of the body; and when these two weights are thus equal, the body will be in equilibrium, if its centre of gravity and that of the fluid displaced, or
32. the centre of buoyancy, are situated in the same vertical. With respect to homogeneous bodies, the centre of buoyancy coincides with that of the immersed part of the body; and that the weight of the fluid displaced may be equal to that of the body, it is necessary that the densities should be in the inverse ratio of the

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Calling these tenths of an inch, we have, by dividing by 10 and by 12, 10800 feet for the absolute length of the circumference; whence,

$$2\,\pi \text{ or } 6{,}2832 : 1 :: 10800 : 1719.$$

In the best spirit levels, the requisite curvature is effected and rendered true by grinding with emery.

bulks, or that the bulk of the immersed part should be to the entire bulk of the body, as its density is to that of the fluid; it 427. follows, therefore, that the determination of the positions of equilibrium of a homogeneous body, placed on the surface of a fluid of a given density greater than that of the body, is reduced to a problem of pure geometry which may be very simply stated. It is required to cut the body by a plane in such a manner, that the bulk of one of its segments shall be to that of the whole body in a given ratio, and that the centre of gravity of this segment and that of the body shall be situated in the same perpendicular to the cutting plane.

439. There are different kinds of equilibrium depending upon the form and position of the floating body. With respect to the sphere, for example, provided its density be less than that of the fluid, it will remain in equilibrium in any position whatever, since the centre of gravity and that of buoyancy continue to be in the same vertical. This will be the case, also, with respect to solids of revolution generally, on the supposition that the axis remains horizontal. Such an equilibrium is called an *equilibrium of indifference.* But when, from the form of the solid or its relative density, it tends, upon being inclined a little, to return to its position, the equilibrium is said to be *stable.* On the other hand, if its tendency after a slight inclination is to depart from its first position, the equilibrium is denominated *unstable.*

440. With respect to the different positions of equilibrium of the same solid, there is a remarkable property which may be demonstrated independently of any calculation. Let us suppose that the body in question is made to turn about a movable axis which is kept constantly parallel to a fixed and horizontal straight line, and that it is made to pass in this way successively through all its positions of equilibrium in which the axis has this direction; we say that the positions of stable and unstable equilibrium will succeed each other alternately, so that if the body be moved from a position of stable equilibrium, the next position will be unstable, the third stable, and so on till it has returned to its first position.

Indeed, while the body is yet very near its first position, it will tend to return to it, this position being supposed to be stable;

but the tendency thus to return will gradually diminish as it revolves, till after a time the body will incline the other way, but before this tendency changes its sign (to borrow an expression from algebra), there will be a position in which it will be nothing, and in which the body will neither incline to return to its first position, nor to depart from it; this, therefore, will be its second position of equilibrium. Now we see that within this part of its revolution, the body tends to return to its first position, and consequently to depart from the second. Beyond this point, the body tends to depart from its first position, and at the same time from the second; therefore the second position of equilibrium is not stable, since on each side of it the body tends to depart from it. Upon its passing this position, its tendency to depart from it diminishes continually, till it becomes nothing; and beyond this the body tends to return toward its second position. The point where this tendency is nothing, is a third position of equilibrium, which is evidently stable; for on each side of it, the body tends to return to it, either approaching toward or receding from its second position. If the third position is stable, it may be shown by the same kind of reasoning that the fourth is not, and that the fifth is, and so on.

Thus, when the body returns to its first position, it will have necessarily passed through an even number of positions of equilibrium, alternately stable and unstable.

441. It is important to be able to distinguish a stable position of equilibrium in a floating body from one which is not so. In order to this let us suppose a body which admits of being divided
Fig. 214. ed by a vertical plane *HFI* into two parts perfectly similar, both as to form and density. Let us suppose, moreover, that this body is made to depart from its position of equilibrium, in such a manner that this section *HFI* remains vertical, and that after having thus disturbed it, we leave it to itself without impressing upon it any velocity; in this way the section *HFI* will remain in the same vertical plane, during the whole motion of the body, for the two portions being perfectly similar in all respects, there is no reason why it should ever depart from the vertical plane in which it was supposed to be first situated. For the same reason, the centre of buoyancy will always be in the section *HFI*, as

well as the centre of gravity. Let G, then, be the centre of gravity; and the position being that of equilibrium, let B be the centre of buoyancy, and HI the intersection of the level of the fluid with the plane HFI, or the *water-line;* in this position, the straight line GB, which connects the two centres, is vertical, and consequently perpendicular to the straight line HI; it inclines generally when the body is made to depart from this position, and at the same time, the centre of buoyancy, and the water-line, change their position upon the plane HFI. I will suppose, therefore, that this centre is the point B', and this line the straight line $H'I'$, when the equilibrium has been disturbed; the forces which will tend to put the body in motion are the weight of the body which is directed according to the virtical GV drawn through the centre of gravity G, and the resultant of the vertical pressures of the fluid upon the surface of the body; this resultant is the buoyancy of the fluid, and is equal to the weight of
the fluid displaced, and is exerted at the point B' its centre of 136.
gravity, in the direction contrary to that of gravity, or according to the vertical $B'Z$. This vertical and the inclined straight line GB being in the same plane, will cut each other in a certain point M called the *metacentre.* It is on the position of this point with respect to the centre of gravity G, that the stability of the
equilibrium depends. The point M may be taken for the point 43.
of application of the buoyancy of the fluid, which will then be exerted according to the straight line MZ; the body will therefore be acted upon by two parallel and contrary forces, applied at the extremities of the straight line GM. It is now proposed to determine in what direction the body will move, and whether these forces will tend to restore it to its position of equilibrium, or to make it depart further from this position.

442. In the first place, if they be unequal, they will produce a
motion of oscillation in the point G. For the centre of gravity 124.
ought to move just as if the two forces were applied directly at this point; therefore, the initial velocity being nothing, its motion will be in a vertical straight line, and in point of magnitude equal to the excess of the greater of the two forces over the less. If at the commencement of the motion, the weight of the body exceeds the buoyancy of the fluid, the point G will begin to descend; its motion will at first be accelerated, but according as the body sinks

in the fluid, the portion displayed will be greater and greater, and, consequently the buoyant effort will increase, till at length it will become equal to the weight of the body. The point G will still continue to move on in the same direction by virtue of its acquired velocity, but then the buoyancy of the fluid exceeding the weight of the body, its velocity will be retarded continually, till finally the downward motion of G will cease, and then it will begin to return toward its first position, and thus it will continue to oscillate till the motion is entirely destroyed by the resistance of the fluid. The extent of these oscillations will be smaller, according to the difference at the outset between the weight of the body and that of the fluid displaced, compared with the weight of the body. If the body be but little removed from its position of equilibrium, this difference will be small, the extent of the oscillations will consequently be small, and will not materially affect the stability of the body.

443. During these oscillations of the point G, the body will turn about this centre, in precisely the same manner as if it were
137. fixed; its motion of rotation will be produced, therefore, by the buoyancy of the fluid, which acts at the point M according to the direction MZ; and its state of equilibrium will be stable or unstable, according as the straight line GB tends to approach to, or recede from, the vertical. Now it is evident from inspection, that the buoyancy of the fluid will tend to restore the straight line GB to its vertical position, whenever the point M is above the point G; on the other hand, if M or the metacentre is below the point G, as at M', for instance, the buoyancy of the fluid, which will then be exerted according to $M'Z'$, will cause the straight line GB to depart further from a vertical position, and tend to upset the floating body. Therefore, when the metacentre is below the centre of gravity, the equilibrium is unstable; and, on the other hand, when the metacentre is above the centre of gravity, the equilibrium is stable; at least with respect to all positions in which the plane HFI continues vertical. If, in a particular case, the metacentre coincides with the centre of gravity, there will be no tendency to turn one way or the other, and the straight line GB will remain stationary, whatever inclination be given to it.

444. When the form of the floating body is known, on the supposition that its position is very near that of equilibrium, it will be easy to determine the place of the metacentre, or rather to determine whether this point is above or below the centre of gravity of the body. Let us suppose, for instance, that this body is a homogeneous horizontal cylinder of an elliptical base, and of half the density of the fluid; let $HFIA$ be a vertical section made at equal distances from the two bases; in the position of equilibrium, one of the two axes will be vertical; and as half of the bulk will be immersed in the fluid, it follows that the other axis will coincide with the *water-line*, and will represent the level of the fluid. The vertical axis AF is the transverse in figure 215, and the conjugate in figure 216. Now we say that in the first case, the metacentre is below the centre G of the ellipse, which is also the centre of gravity of the cylinder, and that it is above it in the second case. Fig. 215, 216.

Draw through the point G, a straight line $A'F'$ making a very small angle with AF; let us now suppose that the axis AF is inclined till $A'F'$ becomes vertical, and that at the same time we raise or depress a very little the centre of gravity G, so that the level of the fluid shall become the straight line $H'I'$, perpendicular to the straight line $A'F'$ at the point G'. In this position, the part $H'F'I'$ of the ellipse $HFIA$ will be immersed in the fluid; and this part is divided into two unequal portions $H'F'G'$ and $I'F'G'$ by the straight line $G'F'$. Now it is evident that the centre of buoyancy will be found in some point B', situated in the greater of these two portions, whence it is evident by looking at the two figures, that $B'M$, parallel to the straight line $F'A'$, will meet the straight line FA at the point M, below the centre of gravity in the first figure, and above it in the second.

Thus the cylinder which we are considering is in a stable or unstable position of equilibrium, according as the conjugate or transverse axis of its base is vertical. Supposing the body to turn about the horizontal straight line which joins the centres of the two bases, it will pass successively through four positions of equilibrium, which will be alternately stable and unstable, agreeably to the general position already advanced.

445. To fix, in a few simple cases, the dimensions of the solid and its relative density to that of the fluid required for a particular state of equilibrium, let the body in question be a homogeneous parallelopiped, placed vertically in the fluid; and let
Fig. 217. DFE be a section of this body through the axis parallel to one of its faces. The solid will evidently sink till the immersed part
427. NF shall be to the whole height AF, as its density is to the density of the fluid; and its centre of gravity and that of buoyancy will be G and B, the middle points respectively of the axis and of the depressed portion NF. Suppose now that the body is inclined a little, shifting its water-line from the position HNI to $H'NI'$, the centre of buoyancy, changing from B to B', will describe a small arc of a circle, which for the extent under consideration may be regarded as a straight line, and B' will be raised by a quantity which will be to the altitude PO of the centre of gravity of the triangle INI', as the area of the rectangle NIF is to that of the triangle INI', that is, as $NF : \frac{1}{2} II'$. Moreover the horizontal motion of B will be to NP or $\frac{2}{3} NI$ in the same proportion. Whence

$$NF : \tfrac{1}{2}II' :: \tfrac{2}{3} NI : BB' = \frac{NI \times II'}{3\,NF}.$$

But, by similar triangles,

$$BB' \text{ or } \frac{NI \times II'}{3\,NF} : BM :: II' : NI;$$

accordingly we have, for the height of the metacentre above the centre of buoyancy,

$$BM = \frac{\overline{NI}^2}{3\,NF} = \frac{\overline{HI}^2}{12\,NF}.$$

Let AF, the height of the parallelopiped, be denoted by h, its breadth or thickness HI by a, and its density or specific gravity by Δ. When the metacentre coincides with the centre of gravity, and the solid floats indifferently in any position, BM is equal to BG or to $\frac{AF - NF}{2}$; that is, (water being the fluid in question,) since 1, the density of the fluid, is to Δ, the density of the solid, as AF or h is to NF or $\Delta\, h$, we shall have

$$\frac{a^2}{12 \triangle h} = \frac{h - \triangle h}{2}.$$

$$2 a^2 = 12 \triangle h^2 - 12 \triangle^2 h^2,$$

or

$$\triangle^2 - \triangle = - \frac{a^2}{6 h^2}.$$

This being resolved after the manner of an equation of the second degree, gives

$$\triangle = \tfrac{1}{2} \pm \sqrt{\frac{3 h^2 - 2 a^2}{12 h^2}}.$$

If the parallelopiped become a cube, then $a = h$, and we have

$$\triangle = \tfrac{1}{2} \pm \sqrt{\frac{1}{12}} = \tfrac{1}{2} \pm \sqrt{\frac{3}{36}} = \tfrac{1}{2} \pm \frac{1{,}732}{6} = 0{,}5 \pm 0{,}29 \text{ near-}$$

ly; that is, the two densities are 0,79 and 0,21 nearly. Between these limits there can be no stability; but above 0,79 or below 0,21 the equilibrium becomes more and more permanent. Hence a cube of beach will float erect in water, while one of fir or cork will overset; yet all these three cubes will stand firmly when placed upon the surface of mercury. We restrict ourselves, in this illustration, to cubes, because we cannot apply the remark to parallelopipeds generally. A stable equilibrium depends, as will be inferred from what has been said, not only upon the relative densities of the solid and fluid, but also upon the proportion between the horizontal and vertical dimensions of the solid. In order to ascertain this proportion in the case of parallelopipeds, and on the supposition of a density equal to half that of the fluid, we have only to put equal to zero the radical part of the above general formula, and we shall have

$$3 h^2 = 2 a^2;$$

accordingly

$$\frac{h^2}{a^2} = \tfrac{2}{3} = \tfrac{16}{24}, \text{ and } \frac{h}{a} = \tfrac{4}{5} \text{ nearly.}$$

Whence, approximatively,

Mech. 42

$$12\,h = 10\,a,$$

or

$$h : a :: 10 : 12,$$

that is, a parallelopiped of half the density of the fluid, and having its height to one side of a square base, as 10 to 12, would float indifferently.

446. But if the relative density of the parallelopiped were either greater or less than $\frac{1}{2}$, its equilibrium would become stable. Thus, if we suppose Δ equal to $\frac{1}{3}$, we shall have the distance of the metacentre above the centre of buoyancy, as follows, namely,

$$BM = \frac{a^2}{12\,\frac{1}{3}\,h} = \frac{12^2}{12 \times \frac{1}{3} \times 10} = \tfrac{36}{10} = 3{,}6 \text{ inches},$$

and for the distance of the centre of gravity above the centre of buoyancy,

$$BG = \frac{h - \frac{1}{3}\,h}{2} = \tfrac{1}{3}\,h = \tfrac{1}{3}\,10 = 3{,}3 \text{ inches};$$

so that the centre of gravity is about 0,3 of an inch below the metacentre. Therefore the equilibrium would be stable.

In like manner, if we substitute $\frac{2}{3}$ instead of $\frac{1}{3}$ in the above equations, or, which is the same thing, take half of each of the above results, we shall have $\frac{1}{2}$ of 0,3, or 0,15 of an inch, for the distance of the centre of gravity below the metacentre.

447. These principles are well illustrated by the masses of ice which appear on the rivers of the colder climates at the opening of spring. Being ordinarily much broader than they are thick, they have a stable equilibrium in their natural position with their broad surface horizontal. But when by striking against each other, or by passing over a fall, they are thrown up sidewise, their equilibrium becomes unstable, and they soon return to their former position. Moreover a piece of ice of a cubical form will still preserve its balance, since its specific gravity does not come within the limits already pointed out, of an unstable equilibrium.*

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* The specific gravity of ice is 0,92, or, compared with sea-water as unity, 0,89.

The ice-bergs that float from the polar seas down into warmer regions, are gradually dissolved, not only by the sun's rays, but also by the currents of warm air and warm water to which they are exposed. But these causes operate more powerfully on the sides than upon the top and bottom, and their horizontal dimensions are thus reduced faster than their vertical, whereby they become unstable, and are overturned. Being still subject to the same kind of influence, they are liable to repeated and frequent changes of position before they are completely wasted.

448. To investigate generally the conditions of equilibrium of a floating body, let $HAIF$ represent a vertical section, the point G of the principal axis ANF being the centre of gravity of the whole, and the point B the centre of buoyancy. The solid being inclined a little, the water-line HNI shifts to $H'NI'$, and the centre of buoyancy B to B'. From what has been said, it will be perceived that the area of HFI is to the sum of the two triangles NII', HNH', as NP is to BB'; that is, Fig. 218. 445.

$$\text{area } HFI : NI \times II' : : \tfrac{2}{3}\, NI : BB',$$

but the triangles INI', BMB', being similar,

$$II' : NI : : BB' : BM,$$

therefore, by multiplying the terms in order, and suppressing the common factor in each of the ratios, we have

$$\text{area } HFI : \overline{NI}^2 : : \tfrac{2}{3}\, NI : BM,$$

or, calling the area HFI, σ, and the length of the water-line, HNI, a,

$$\sigma : (\tfrac{1}{2}\, a)^2 : : \tfrac{1}{3}\, a : BM = \frac{a^3}{12\,\sigma}.$$

Accordingly, the equilibrium will be stable when the cube of the length of the water-line NI, divided by 12 times the area of the section, exceeds the interval BG between the centre of gravity and that of buoyancy. If this quotient be just equal to BG, the equilibrium will be that of indifference; and lastly, when this quotient is less than BG, the equilibrium will be unstable, and the body will be liable upon a slight inclination to overset.

Whatever be the figure of the section *HFI*, its area and consequently the centre of gravity and that of buoyancy may be found to any required degree of exactness by the method of article 114.

Although the above formula has reference only to a single lamina, and to motion in the plane of this lamina, it is still applicable to any solid whose parallel sections are equal and similar, for in this case the whole may be considered with respect to motion in a parallel plane, as concentrated in the middle section or lamina represented by *HFI*; and with respect to motion in a vertical plane perpendicular to the lamina, by supposing a corresponding section, and putting σ' equal to the area of this section, and a' equal to the length of the water-line, we shall have the same formula to express the conditions of equilibrium as before.

When the sections or laminæ are unequal, we find the height of the metacentre of each lamina, and multiply it by the bulk of this lamina, and then divide the sum of the products, or moments of the several laminæ, by the sum of the laminæ for the height of
76. the common metacentre.

449. In the case of a merchant ship, it will furnish a tolerable approximation to take the section near the prow where the girth is commonly the largest. The transverse section of the hull of a ship is not materially different from the form of a parabola. Therefore, on this supposition, the height of the metacentre above
Fig. 219. the centre of buoyancy, or *BM*, is equal to the cube of *HI*, the
Cal. 94. length of the water-line, divided by twelve times the area of *HFI*. But the area *HFI* is equal to $\frac{2}{3}$ $HI \times NF$. Hence,

$$BM* = \frac{\overline{HI}^3}{12\,\sigma} = \frac{\overline{HI}^3}{12 \times \frac{2}{3}\, HI \times NF} = \frac{\overline{HI}^2}{8\,NF} = \frac{\overline{NI}^2}{2\,NF}.$$

* Where great accuracy is required, the following formula may be used; namely,

$$BM = \frac{\frac{2}{3} \int \overline{NI}^3 \times e}{b},$$

that is, BM is equal to half the parameter of the parabola. B being the centre of gravity of the parabola HFI, its height is readily found to be $\frac{3}{5}$ NF. Therefore for the whole height of the metacentre above the keel, we have Trig. 172. 107.

$$\tfrac{3}{5}\,NF + \frac{\overline{NI}^2}{2\,NF} = \frac{6\,\overline{NF}^2 + 5\,\overline{NI}^2}{10\,NF}.$$

Such is the height of the metacentre above the keel, on the supposition that the vertical sections are all equal and parabolic; which is nearly the case with respect to long track-boats. But the figure of the keel in most vessels, fitted for sailing, approaches to a semi-ellipse, which is likewise the general form of a horizontal section. Owing to these modifications, the metacentre is found to be depressed about one fourth part, and consequently its height above the centre of buoyancy will be

$$\tfrac{3}{4} \times \frac{\overline{HI}^2}{8\,NF} = \frac{3\,\overline{NI}^2}{8\,NF}.$$

In a ship, for example, whose water-line is 40 feet, and the depth of its immersed portion 15 feet, we shall have for the height of the metacentre above the centre of buoyancy,

$$\frac{3\ (20)^2}{8 \times 15} = \tfrac{1200}{120} = 10 \text{ feet}.$$

But the centre of gravity of the immersed part is $\frac{2}{5}$ 15 or 6 feet below the water-line. Hence the metacentre is 4 feet above the water-line. The ship will therefore float securely so long as the general centre of gravity is kept under that limit. In

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in which $\int \overline{NI}^3$ represents the sum of the cubes of the perpendiculars RQ, ON, &c., of figure 50, these perpendiculars being taken at equal distances and so near to each other that the included portions of the curve ON, OK, &c., may be considered as straight lines, the common distance being denoted by c, and the bulk of the immersed part of the vessel by b. The investigation of this formula is very simple, and is omitted here merely on account of its length. See Bézout's Mécanique, art. 359.

loaded vessels, the centre of gravity has commonly been found to be higher than the centre of buoyancy, by about the eighth part of the extreme breadth. Accordingly, in the present instance, the centre of gravity of the whole mass would still be one foot below the surface of the water, or five feet lower than the metacentre, which would be amply sufficient for the stability of the ship.

450. Such is the position of the metacentre in the vertical plane at right angles to the longitudinal axis, and which regulates the *rolling* of a vessel from side to side. But there is another similar point in the plane of the masts and keel, which determines the *pitching*, or the movement of alternate rising and sinking of the prow. The height of this metacentre is derived from the same formula, by substituting only the length, for the breadth of the vessel. Thus, let the keel measure 180 feet, and we have

$$\frac{3\,(90)^2}{8.15} = 202\tfrac{1}{2} \text{ feet.}$$

With such a strong tendency to stability, therefore, in the direction of its course, a ship can scarcely ever founder in consequence of pitching at sea.

The formula now given for computing the height of the metacentre above the centre of buoyancy, may, with some modification, be deemed sufficiently accurate in practice. It is best adapted, however, for cutters or frigates, and will require to be somewhat diminished in the case of merchant vessels. Mr. Atwood performed a laborious calculation on the hull of the Cuffnells, a ship built for the service of the East India Company, having divided it into 34 transverse sections, of five feet interval. The result was, that the metacentre stood only 4 feet 3 inches above the centre of buoyancy. But that ship, being designed chiefly for burthen, appears from the drawings to have been constructed after a very heavy model, its vertical sections approaching much nearer to rectangles than parabolas. To suit it, the formula above given would have required to be reduced two thirds, or to $\frac{\overline{NI}^2}{4\,NF}$. Now the breadth of the principal section was

43 feet and two inches, and its depth 22 feet 9 inches. Whence $\frac{(21,6)^2}{91} = 5,1$ feet, differing little from the conclusion of a stricter but very tedious process.

451. Since the height of the metacentre is inversely as the draught of a vessel, and directly as the square of its breadth, its stability depends mainly on its spreading shape. This property is an essential condition in the construction of life-boats. But the lowering even of the centre of gravity has been found to be sometimes insufficient to procure stability to new ships, which, after various ineffectual attempts, were rendered serviceable, by applying a sheathing of light wood along the outside, and thus widening the plane of floating.

452. It is not very difficult to determine the centre of buoyancy, by gauging the immersed part of the hull. A cubic foot of sea-water weighs 64$^{\text{lb.}}$ avoirdupois, and 35 feet, therefore, make a ton. The load of the vessel corresponding to every draught of water may be hence computed.

453. The height of the metacentre above the centre of gravity in a loaded vessel, may by determined by simple observation. Fig. 220.
Let a long, stiff, and light beam be projected transversely from the middle of the deck, and a heavy weight suspended from its remote end, inclining the ship to a certain angle, which is easily measured. Thus, if NL represent this lever, q the weight attached, M the metacentre, and GMQ the inclination produced, G being the centre of gravity, and GR a perpendicular drawn from it to the vertical $L\,q$, the power of the weight q to incline the vessel will be expressed by $q \times GR$; but p denoting the entire weight of the vessel, the effort exerted at the metacentre to keep the mast erect, will be represented by

$$p \times GQ, \text{ or } p \times GM \times \sin GMQ.$$

Wherefore

$$q \times GR = p \times GM \times \sin GMQ,$$

and consequently the elevation GM above the centre of gravity is expressed by $\frac{q}{p} \cdot \frac{GR}{\sin GMQ}$. Now GR may, without any sen-

sible error, be assumed as equal to the length LN of the beam from the middle of the deck. Supposing the height of the metacentre to be 3 feet $10\frac{1}{2}$ inches above the centre of gravity, a weight equal to the two hundredth part of the burthen or tonnage of the ship, and acting on a lever of 50 feet in length, would occasion an inclination of five degrees. If the experiment were performed in a wet-dock, or on a smooth calm sea, such a small angle could be measured with sufficient accuracy. In calculating the effect of this disturbing influence, it is easy to perceive that half the weight of the beam should be added to q. A trifling correction may be likewise made, for assuming GR as equal to NL; by first diminishing NL, by its product into the versed sine of the inclination, and next augmenting it, by the product of GN into the sine of that angle.

A similar method might be adopted to discover the height of the longitudinal metacentre of the ship, above the common centre of gravity. But, acting in this direction, a greater load will be required to produce a sensible depression. Let such a load be carried to the prow of the vessel, and again transferred to the stern. The intermediate place of the centre of gravity is hence determined, for its distances from these opposite points of pressure must evidently be inversely as the corresponding angles of inclination. The small change of the centre of gravity occasioned by the interchange of these loads, may likewise be computed. Finally, therefore, the product of either load into its distance from the centre of gravity, being divided by the product of the whole burthen of the ship into the sine of the inclination, will give the height of the metacentre of the longitudinal section on which depends the motion of pitching.

Capillary Attraction.

454. The most curious natural phenomena are those which make us acquainted with the intimate constitution of bodies

and the reciprocal action which their particles exert upon each other. We come now to consider a class of these phenomena of considerable extent and variety, and which are the more deserving of attention, as they are susceptible of a rigorous calculation.

If a disk of glass, marble, or metal, &c., be suspended to the scale of a balance, and counterpoised by an equal weight in the opposite scale, upon being made to touch the surface of a liquid capable of moistening it, it will be found to adhere with a certain force, and to require an additional weight in the opposite scale of the balance to detach it. This adhesion is not produced by the pressure of the air, for it takes place equally well in a vacuum. We infer, therefore, that it is the particles of the solid which attach themselves to the particles of the fluid by virtue of a *force of affinity.* But there is to be inferred also a similar action between the particles of the fluid itself. Indeed when the disk is capable of being moistened by the liquid, as is the case when glass is used with water or alcohol, the disk, upon being withdrawn brings with it a small liquid film, or lamina, which adheres to it. It is not then, strictly speaking, the solid which is detached from the liquid, it is this small lamina which is separated from the particles immediately below it. Now the force employed thus to detach it, is incomparably more considerable than the proper weight of this lamina; consequently the excess of force proves the existence of an internal adhesion in the liquid which would keep the small lamina united to the rest of the liquid mass independently of gravity.

According to the notions which we have formed of the reciprocal action of the particles of bodies upon each other, the force in question seems to be of the same nature and to have a sensible effect only at very small distances. This is moreover proved by experiment. Whatever be the thickness of the disk, so long as the form and substance are the same, the force required to detach it from a given liquid, is also the same. Accordingly, beyond a certain thickness, probably less than any within the reach of human art to attain, any augmentation has no effect capable of being appreciated. Whence it will be seen that this action is not capable of producing sensible effects, ex-

cept at distances extremely small. But as a further proof, it may be mentioned that all disks of the same size, whatever be the substance, provided it is capable of being moistened by the liquid, require precisely the same force to detach them; so that in these cases the thin film of water, which attaches itself to their surfaces, places these surfaces and the rest of the fluid at intervals sufficiently great to prevent any sensible action taking place; and the force required to detach all disks of the same size, whatever be the substance, is precisely the same, since it is that which is necessary to detach the liquid from itself.

455. Phenomena arising from the same cause, but differing
Fig.221. in appearance, are also observed when tubes of a small bore are immersed in a liquid. If the liquid is of a nature to moisten the tube, it will be found to ascend into the interior, and to maintain itself above the natural level. When glass tubes, for example, of a fine bore are immersed endwise in water or alcohol, this elevation of the fluid will take place; and in these cases, the upper extremity of the column is concave. But if the liquid is not of a nature to moisten the tube, as is the case with mercury, melted lead, &c., used with glass taken in its ordinary state, the liquid in the tube will be depressed instead of being elevated, and the upper extremity of the column will be convex. In all these cases the elevation or depression is the more considerable according as the diameter of the bore is less. Such are the phenomena which are called capillary from the circumstance of the fineness of the bore of the tube.

The phenomena being the same in a vacuum as in the open air, they are not connected with the pressure of the atmosphere. *

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* In the discussion formerly maintained upon this subject, a perplexing fact was stated; namely, that if a glass tube consisting of two cylinders of different bores, joined endwise, be immersed in water, the larger end being downward, so as to cause the fluid to rise into the smaller part of the tube, the column sustained will be of the same length as in a tube whose bore is throughout of the same size with this smaller. The experiment is now found not to succeed in a vacuum. The peculiarity of the phenomenon, therefore, must depend upon the pressure of the atmosphere.

But they depend, like the preceding, upon the attraction exerted by the tube upon the liquid and by the liquid upon itself; so that when the thickness of the tube is made to vary, the bore remaining unchanged, the elevations and depressions of the liquid remain the same, which proves that beyond a certain thickness, probably too small for us to attain, any additional matter that may be accumulated will have no appreciable effect. It follows from this law, that when tubes of the same diameter are completely moistened throughout by the liquid, the elevation or depression will be the same in all, whatever the substance of the tube, which shows that the thin film attached to the interior surface, removes by its interposition the rest of the liquid mass so as to render the attraction of the tube insensible; consequently the elevation is the same in all tubes of the same bore, because it is equal to that which would proceed from a tube of the same diameter formed of the liquid itself.*

456. Setting out from the results furnished by the calculus, we are able to give a satisfactory explanation of the phenomena of capillary tubes. Beginning with the case in which the fluid is elevated above the natural level, and which requires the upper extremity of the fluid column to be concave, we suppose an infinitely small filament of fluid extending from the lowest point Fig. 222.
of the meniscus along the axis of the tube, and then returning in any manner through the mass of the liquid to the free surface. The fluid being in a state of equilibrium, this filament will be in

* The diameter of the bore of a tube is found by first weighing the tube empty, and then after having introduced a certain quantity of mercury, weighing it again. The excess of the latter weight above the former will be the weight of the column of mercury. By calling this weight w, the length of the column l, and the radius of the bore R, π being the ratio of the circumference of a circle to its Geom. 291.
diameter, we shall have for the bulk of mercury contained in the tube $\pi R^2 l$. If w' be the weight of a cubic inch of mercury at the temperature assumed in the experiment, and R and l be also expressed in inches, or parts of an inch, $w' \pi R^2 l$ will be the weight of the column in question; whence

$$w' \pi R^2 l = w, \text{ and } R = \sqrt{\frac{w}{w' \pi l}}.$$

a state of equilibrium. But it is pressed downward at the two extremities with unequal forces. The force exerted at the free surface is the action of a body terminated by a plane surface; the other in the interior of the tube is the action of the same body terminated by a concave surface, or one in which there is a contrary attraction upward, the little annulus cut off by a horizontal plane passing through the lowest point of the meniscus, and which is supported by the attraction of the glass, exerting an upward force. It is necessary, therefore, in order that an equilibrium may take place that the fluid should rise in the tube till the weight of the column thus elevated above the natural level should compensate for this difference in the downward pressures exerted at the two extremities of the filament. This difference is in the inverse ratio of the diameter of the tube; the height of the small column must accordingly be in the same ratio; and this is conformable to the results of our observation.

457. The heights to which water and alcohol ascend in capillary tubes were observed by M. Gay Lussac with the greatest care. The following are a few of his results.

Water.

	Diameter of the tube.	Height to the lowest point of the concavity.	Temperature.
(1.)	1,29441*	23,1634	47,5° Fah.
(2.)	1,90381	15,5861	47,5°.

Alcohol, (specific gravity being 0,81961.)

	Diameter of the tube.	Height to the lowest point of the concavity.	Temperature.
(1.)	1,29444	9,18235	47,5°
(2.)	1,90381	6,08397	47,5°.

M. Gay Lussac measured also the ascent of water between two plates of glass ground perfectly plane, and placed exactly par-

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* The measures of M. Gay Lussac are given in millimetres or 0,039371 of an inch. The results in which we are principally concerned, are reduced to English inches.

allel to each other. The result of his observations was as follows. Fig.224.

Distance of the plates.	Height to the lowest point of the concavity.	Temperature.
1,069	13,574	62°.

458. Let AB be a vertical tube whose sides are perpendicular to its base, and which is immersed in a fluid that rises in the interior of the tube above its natural level. A thin film of fluid is first raised by the action of the sides of the tube; this film raises a second film, and this second a third, till the weight of the volume of fluid raised exactly balances all the forces by which it is actuated. Hence it is obvious, that the elevation of the column is produced by the attraction of the tube for the fluid, and the attraction of the fluid for itself. Let us suppose that the inner surface of the tube AB is prolonged to E, and after bending itself horizontally in the direction ED, that it assumes a vertical direction DC; and let us suppose the sides of this tube to be formed of a film of ice, or to be so extremely thin, as not to have any action on the fluid which it contains, and not to prevent the reciprocal action which takes place between the particles of the first tube AB and the particles of the fluid. Now, since the fluid in the tubes AE, CD, is in equilibrium, it is obvious, that the excess of pressure of the fluid in AE is destroyed by the vertical attraction of the tube and of the fluid upon the fluid contained in AB. In analysing these different attractions, Laplace considers first those which take place under the tube AB. The fluid column BE is attracted, (1.) by itself; (2.) by the fluid surrounding the tube BE. But these two attractions are destroyed by the similar attraction experienced by the fluid contained in the branch DC, so that they may be entirely neglected. The fluid in BE is also attracted vertically upward by the fluid in AB; but this attraction is destroyed by the attraction which the fluid in BE exerts in turn upon that in AB, so that these balanced attractions may likewise be neglected. The fluid in BE is likewise attracted vertically upwards by the tube AB, with a force which we shall call q, and which contributes to destroy the excess of pressure exerted upon it by the column BF, raised in the tube above its natural level. Fig.222.

Now the fluid in the lower part of the round tube AB is attracted, (1.) by itself; but the reciprocal attractions of a body do not communicate to it any motion, if it is solid, and we may, without disturbing the equilibrium, conceive the fluid in AB frozen. (2.) The fluid in the lower part of AB is attracted by the fluid within the tube BE; but as the fluid of the tube BE is attracted upwards by the same force, these two actions may be neglected as balancing each other. (3.) The fluid in the lower part of AB is attracted by the fluid which surrounds the ideal tube BE, and the result of this attraction is a vertical force acting downwards, which we may call $-q'$, the contrary sign being applied, as the force is here opposite to the other force q. As it is highly probable that the attractive forces exerted by the glass and the water vary according to the same function of the distance, so as to differ only in their magnitude, we may employ the constant coefficients p, p', as measures of their intensity, so that the forces q, $-q'$, will be proportional to p, p'; for the interior surface of the fluid which surrounds the tube BE, is the same as the interior surface of the tube AB. Consequently, the two masses, namely, the glass in AB, and the fluid around BE, differ only in their thickness; but as the attraction of both these masses is insensible at sensible distances, the difference of their thicknesses, provided their thicknesses are sensible, will produce no difference in the attractions. (4.) The fluid in the tube AB is also acted upon by another force, namely, by the sides of the tube AB in which it is enclosed. If we conceive the column FB divided into an infinite number of elementary vertical columns, and if, at the upper extremity of one of these columns, we draw a horizontal plane, the portion of the tube comprehended between this plane and the level surface BC of the fluid, will not produce any vertical force upon the column; consequently, the only effective vertical force is that which is produced by the ring of the
35. tube immediately above the horizontal plane. Now the vertical attraction of this part of the tube upon BE, will be equal to that of the entire tube upon the column BE, which is equal in diameter, and similarly placed. This new force will therefore be represented by $+q$. In combining these different forces, it is manifest that the fluid column BF is attracted upwards by the two forces $+q$, $+q$, and downwards by the force $-q'$;

consequently, the force with which it is elevated will be $2q - q'$. If we represent the bulk of the column BF by b, its density by Δ, and the force of gravity by g, then $g \Delta b$ will represent the weight of the elevated column; but, as this weight is in equilibrium with the forces by which it is raised, we shall have the following equation;

$$g \Delta b = 2q - q'.$$

If the force $2q$ is less than q', then b will be negative, and the fluid will be depressed in the tube; but as long as $2q$ is greater than q', b will be positive, and the fluid will rise above its natural level.

Since the attractive forces, both of the glass and fluid, are insensible at sensible distances, the surface of the tube AB will act sensibly only on the film of fluid immediately in contact with it. We may therefore neglect the consideration of the curvature, and consider the inner surface as developed upon a plane. The force q will therefore be proportional to the width of this plane, or, which is the same thing, to the interior circumference of the tube. Calling c, therefore, the circumference of the tube, we shall have $q = pc$, p being a constant quantity representing the force of attraction of the tube AB for the fluid, in the case where the attractions of different bodies are expressed by the same function of the distance. In every case, however, p expresses a quantity dependent on the attraction of the matter of the tube, and independent of its figure and magnitude. In like manner we shall have $q' = p'c$; p' expressing the same thing with regard to the attraction of the fluid for itself, that p expresses with regard to the attraction of the tube for the fluid. By substituting these values of q, q', in the preceding equation, we shall have

$$g \Delta b = 2pc - p'c = c(2p - p') \qquad \text{(1.)}$$

If we now substitute, in this general formula, the value of c in terms of the radius, if it is a capillary tube, or in terms of the sides, if the section is a rectangle, and the value of b in terms of the radius and altitude of the fluid column, we shall obtain an equation by which the heights of ascent may be calculated for tubes of all diameters, when the height, belonging to any given diameter, has been ascertained by direct experiment.

In the case of a cylindrical tube, let π represent the ratio of the circumference to the diameter, h the height of the fluid column reckoned from the lowest point of the meniscus, h' the mean height to which the fluid rises, or the height at which the fluid would stand if the meniscus were to settle down and assume a level surface; then we have πR^3 for the solid contents of a
Geom. 555. cylinder of the same height and radius as the meniscus; and as the meniscus, added to the solid contents of a hemisphere of the same radius, must be equal to πR^3, (or in other words, the cylinder πR^3, diminished by the hemisphere $\frac{2}{3} \pi R^3$, is equal to the mensicus,) we have $\pi R^3 - \frac{2 \pi R^3}{3}$, or $\frac{\pi R^3}{3}$, for the solid contents of the meniscus. But since $\frac{\pi R^3}{3} = \pi R^2 \times \frac{R}{3}$, it follows that the meniscus $\frac{\pi R^3}{3}$ is equal to a cylinder whose base is πR^2, and altitude $\frac{R}{3}$. Hence, we have $h' = h + \frac{R}{3}$; or, which is the same thing, the mean altitude h' is always equal to the altitude h of the lower point of the concavity of the meniscus, increased by one third of the radius, or one sixth of the diameter of the capillary tube. Now, since the contour c of the tube $= 2 \pi R$, and since the bulk b of water raised is equal to $h' \times \pi R^2$, we have, by substituting these values in the general formula (I.)

$$g \,\Delta\, h' \pi R^2 = 2 \pi R (2p - p') \qquad \text{(II.)}$$

and, dividing by πR and $g \,\Delta$, we obtain,

$$h' R = 2 \frac{2p - p'}{g \,\Delta}, \text{ and } h' = 2 \frac{2p - p'}{g \,\Delta} \times \frac{1}{R} \qquad \text{(III.)}$$

In applying this formula to M. Gay Lussac's experiments, we have $2 \frac{2p - p'}{g \,\Delta} = R h' = 0{,}647205 \times (23{,}1634 + 0{,}215735)$ $= 15{,}1311$, or 0,023454 of an inch for the first experiment; and, *since the heights are inversely as the radii or diameters*, 0,023454 *or its double* 0,046908 *is a constant quantity.* In order to find the height of the fluid in the second tube by means of this constant quantity, we have

$$\text{R} = \frac{1{,}90381}{2} = 0{,}951905,$$

and

$$2\,\frac{2\,p - p'}{g\,\Delta} \times \frac{1}{\text{R}} = h' = \frac{15{,}1311}{0{,}951905} = 15{,}8956,$$

from which if we subtract one sixth of the diameter, or 0,3173, we have 15,5783 for the altitude h of the lowest point of the concavity of the meniscus, which differs only 0,0078 or 0,0003 of an inch from 15,5861, the observed altitude.

If we apply the same formula to M. Gay Lussac's experiments on alcohol, we shall find for the constant quantity

$$2\,\frac{2\,p - p'}{g\,\Delta} = 6{,}0825,$$

as deduced from the first experiment, and $h = 6{,}0725$, which differs only 0,01147, or 0,00045 of an inch from 6,08397, the observed altitude.

From these examples, it will be seen that the mean altitudes, or the values of h', are reciprocally proportional to the diameters of the tubes very nearly; and that in accurate experiments, the correction made by the addition of the sixth part of the diameter of the tube is indispensably requisite.

459. If the section of the bore in which the fluid ascends is a rectangle, whose greater side is a, and smaller side δ, the base of the elevated column will be $a\,\delta$, and its perimeter $2\,a + 2\,\delta$. Then the meniscus will be equal to the small rectangular prism, whose base is $a\,\delta$ and height $\frac{1}{2}\,\delta$, minus the semicylinder whose radius is $\frac{1}{2}\,\delta$ and length a; accordingly, we have for the solidity of the meniscus, Fig. 224.

$$\frac{a\,\delta^2}{2} - \frac{a\,\pi\,\delta^2}{8} = \frac{a\,\delta^2}{2}\left(1 - \frac{\pi}{4}\right),$$

that is,

$$h' = h + \frac{\delta}{2}\left(1 - \frac{\pi}{4}\right).$$

Hence, if in the general (I.) equation we substitute for c its equal

$$2\,a + 2\,\delta,$$

and for b its equal $a\,\delta\,h'$, we shall have

$$g\,\Delta\,h'\,a\,\delta = (2\,p - p') \times (2\,a + 2\,\delta);$$

and, dividing by a and by $g\,\Delta$, we have

$$\delta\,h' = 2\,\frac{2\,p - p}{g\,\Delta} \times \left(1 + \frac{\delta}{a}\right),$$

and

$$h' = 2 \times \frac{\frac{2\,p - p'}{g\,\Delta} \times \left(1 + \frac{\delta}{a}\right)}{\delta}.$$

In applying this formula to the elevation of water between two glass plates, the side a is very great compared with δ, and therefore the quantity $\frac{\delta}{a}$ being almost insensible, may be safely neglected. Hence the formula becomes

$$h' = 2\,\frac{2\,p - p'}{g\,\Delta} \times \frac{1}{\delta}.$$

By comparing this formula with the formula (III.) it is evident that water will rise to the same height between glass plates, as in a tube, provided the distance δ between the two plates is equal to R, or half the diameter of the tube; in other words, that, when the distance between the plates is equal to the diameter of the tube, the elevation in the former case is half that in the latter. This result was obtained by Newton, and has been confirmed by the experiments of succeeding philosophers.

As the constant quantity $2\,\frac{2\,p - p'}{g\,\Delta}$ is the same as that already found for capillary tubes, we may take its value, namely, 15,1311, and substitute it in the preceding equation; we shall then have

$$h' = \frac{15{,}1311}{1{,}069} = 14{,}1544;$$

and since

$$h = h' - \frac{\delta}{2}\left(1 - \frac{\pi}{4}\right),$$

subtracting

$$\frac{\delta}{2}\left(1-\frac{\pi}{4}\right)=0{,}1147,$$

from h or 14,1544, we have

$$h = 14{,}0397,$$

which differs 0,4657 of a millimetre, or 0,0183 of an inch, from 457.
13,574, the observed altitude.

460. If the plates are inclined to each other at a small angle, the line of meeting being vertical, the water will rise between Fig. 225.
them to different heights according to the general law above enunciated; that is, the distances at L, G, being LN, GI, we shall have

$$GH : LM :: LN : GI :: MO : HK.$$

But, by similar triangles,

$$MO : HK :: FM : FH,$$

whence

$$GH : LM :: FM : FH,$$

that is, the heights at different points of the curve $ELGB$ are inversely as the distances from the line of meeting EF of the plates. Therefore, since the relation of the lines FM, FH, &c., to the lines LM, GH, &c., is the same as that of the abscissas to the ordinates in the common hyperbola, the surface of the fluid between the plates answers to this curve.

461. If the relative attraction of the parts of the fluid for itself, (in the case of tubes for example) and for the substance of the Fig. 223.
tube, be such that the surface of the fluid column in the tube becomes convex, instead of being concave, the effect is precisely the reverse of that above considered; that is, when an equilibrium occurs, the filament occupying the axis of the tube and rising without the tube to the free surface, will have its extremity F depressed; since, instead of an excess of upward attraction proceeding from a sustained annulus, situated above a horizontal plane passing through the extremity F, there will be a deficiency of upward attraction equal to the effect of this same annulus. Accordingly, the de-

pressions will, like the elevations, be inversely as the diameters of the tubes, and the whole theory above given, with this single modification, is strictly applicable. Between plates also, and between concentric tubes, a depression will take place corresponding to the elevation in the case where the upper surface is concave.

It is to be observed, however, that the deficiency of attraction in the case of mercury used in connection with glass, taken in its ordinary state, is to be ascribed to a want of contact between the fluid and the substance of the tube or plate, arising from a film of moisture which ordinarily attaches itself to glass, and which being completely removed, mercury is found to present a concave surface like water, and consequently to rise in a tube and between plates above its natural level.* Indeed water may be made to exhibit the same apparent anomaly, by having the surface of the glass, whether tube or plate, smeared with a thin coat of tallow or wax.

462. The peculiar character of this theory consists in this, that it makes every thing depend upon the form of the surface. The nature of the solid body and that of the fluid determine simply the direction of the first elements, where the fluid touches the solid, for it is at this point only that their mutual attraction is sensibly exerted. These directions being given, they become the same always for the same fluid and the same solid substance, whatever be the figure of the body itself which is composed of this substance. But beyond the first elements and beyond the sphere of action of the solid, the direction of the elements and the form of the surface are determined simply by the action of the fluid upon itself.

We have seen that the elevation of a liquid between parallel plates, is half of that which takes place in tubes whose diameter is

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* Barometer tubes properly cleansed and freed from humidity, by having the mercury repeatedly boiled in them, will exemplify the truth of this remark. Moreover, with the knowledge of this fact, we can readily satisfy ourselves by simple inspection, whether the requisite attention was paid to this particular in the construction of the instrument.

equal to the distance of the plates. The cause which determines this ratio is to be found in the above theory. For, in the case of tubes, the action of the concave or convex surface upon the elevated or depressed column is half of the action of two spheres which have for radii the greatest and least radii of the osculating circles to the surface at the lowest point. The tube being flattened in any direction, the radius of the corresponding curvature augments, and finally becomes infinite, when the flattened sides of the tube become parallel plane surfaces. The first part of the attraction of the surface being inversely as this radius, will become zero, and there will remain only the term depending on the other osculating radius, and the attractive force is accordingly reduced one half. Such is the simple and rigorous result furnished by the theory of Laplace.

463. This theory serves to explain also, and with the same simplicity, all other capillary phenomena. Thus, the ascent of water between concentric tubes, and in conical tubes; the curvature which water assumes when adhering to a glass plate; the spherical form observed in the drops of liquids; the motion of a drop which takes place between plates having a small inclination to each other and to the horizon; the force which causes drops floating on the surface of a liquid to unite; the adhesion of plates to the surface of a liquid, which is in many cases so great as to require a considerable weight to separate them; — these effects, so various, are all deduced from the same formula, not in a vague and conjectural way, but with numerical exactness.

On the apparent Attraction and Repulsion observed in Bodies floating near each other on the Surface of Fluids.

464. (1.) If two light bodies, capable of being wetted, be placed at the distance of one inch from each other on the surface of a basin of water, they will float at rest, and without approaching each other. But if they be placed at the distance of only a small part of an inch, as two or three tenths, they will rush Fig. 226. together with an accelerated motion.

Fig.227. (2.) If the two bodies are of such a nature as not to suffer the fluid to adhere to them, as is the case with balls of iron used in connection with mercury, the same phenomena will be observed.

Fig.228. (3.) But if one of the bodies is susceptible of an adhesion of the fluid, and the other not, as two balls of cork, for example, one of which has been carbonized by the flame of a lamp; the effect will be the reverse of that above stated; that is, the bodies will seem to repel each other, when brought very near together, and with forces similar to those with which in the former ease they tended to unite.

Moreover, a single ball will approach to, or recede from, the side of the vessel, as it would approach to or recede from another ball, according as the substance of the vessel and that of the ball are similar or dissimilar as to their disposition to cause an adhesion of the fluid.

465. In these experiments the approach and recession of the floating bodies are not the effect of a real attraction or repulsion between the bodies; for, if the bodies, instead of being placed upon the surface of a liquid, be suspended by long, slender threads, nothing of the kind is to be perceived. We must therefore look for some other cause to which to refer these appearances.

Fig.229. If two plates of glass AB, CD, be suspended in water parallel to each other, and at such a distance, that the point H, where the two curves of elevated fluid meet, shall be on a level with the common surface, the plates will remain in equilibrium. But on being brought so near to each other, that the point H shall be above the common level of the surface, the mass of fluid thus raised will have the effect of a chain attached at its extremities to the plates, in drawing the plates toward each other. The approach of the balls to each other under similiar circumstances, is to be referred to the same cause.

When the point H is below the general level, on account of a want of adhesion in the parts of the fluid to the plates, the pressure of the plates inward toward each other not being counterbalanced by the pressure in the opposite direction, they must approach each other, and with a greater or less force, according

to the depression of the point H, or the nearness of the plates to each other. This affords an explanation of the second case above stated.

If one of the floating bodies, as A, for example, is susceptible of being wetted, while the other B is not, the fluid will rise around A and be depressed around B. Accordingly, when the balls are near to each other, the depression around B will not be symmetrical, and the body being thus placed as it were upon an inclined plane, its equilibrium will be destroyed, and it will move off from the other body in the direction of the least pressure. Fig. 228.

These phenomena, of which we have given only a familiar explanation, are all comprehended in Laplace's theory of capillary attraction; and the attractive and repulsive forces are capable, on that theory, of being subjected to a rigorous calculation.

Of the Barometer.

466. If we take a glass tube thirty-three or thirty-four inches in length, closed at one extremity and open at the other, and having filled it with mercury, place the finger over the open extremity and thus immerse it in a basin of the same liquid without suffering the air to enter the tube, the mercury will settle down in the tube, leaving a vacuum above it, till its weight is exactly counterbalanced by the pressure of the atmosphere, exerted upon the surface of the mercury in the basin. This instrument is called a *barometer*.* The perpendicular height at which the mercury is ordinarily maintained at the level of the sea, is very nearly thirty inches.

From what has been said of the manner in which the pressure of fluids is propagated, it will be perceived that it is immaterial what be the extent of surface in the basin, or whether the atmospheric pressure be applied at the top, or, by means of a flexible bag containing the liquid, at the bottom and sides. If instead of entering a basin the tube turn up at the bottom, as in figure 230, so as to admit the air at C, the perpendicular elevation

* See note on the construction and history of the barometer.

above a horizontal line coinciding with the surface at E or F, will be the measure of the atmospheric pressure. This elevation, moreover, is independent of the form of the tube and the particular quantity of mercury contained in it. On the suppo-
Fig. 232. sition, however, that the base of the tube is an inch square, the pressure is equal to that of a parallelopiped of mercury 30 inches in length, or, which amounts to the same thing, to the weight of
410. 30 cubic inches of mercury. Now 30 cubic inches of water is equal to $30 \times 252,525$ grains, or 15,78 troy ounces. Whence 30 cubic inches of mercury is equal to $15,78 \times 13,57$ or 214,12 troy ounces; that is, to 234,7 ounces avoirdupois, or to $14,7^{\text{lb.}}$. We infer, therefore, that the pressure of the atmosphere amounts to nearly $15^{\text{lb.}}$ upon every square inch of surface, or to about one ton upon every square foot. A common sized man exposes a surface of 10 or 11 feet, and is consequently subjected to a pressure of as many tons' weight. The entire surface of the earth being estimated at 5575680000000000 feet, this number will express the weight nearly of the whole atmosphere in tons, a certain deduction being made for the space occupied by mountains and elevated regions. This pressure being exerted upon the surface of the ocean, fishes are exposed to it in addition to the weight of their natural element. But the proportion

$$1 : 13,57 :: 30 : 407,1,$$

gives 407,1 inches, or 34 feet nearly, for the length of a column of water equivalent to that of 30 inches of mercury or the pressure of the atmosphere.* Accordingly for every 34 feet depth a pressure is exerted of a ton upon every foot of surface, over and above that arising from the atmosphere. Now fishes are sometimes caught at the depth of 2600 or 2700 feet, where the pressure of the water amounts to nearly 80 atmospheres or 80 tons upon a square foot; yet these fishes are not injured by such an immense weight, or sensibly impeded in their motions. The reason is, that they are filled with fluids, which from their impenetrability oppose a sufficient resistance to this pressure, and thus preserve the most delicate membranes from being ruptured.

* The mean pressure of the atmosphere is more accurately estimated at 29,922 English inches.

With regard to the facility and rapidity of their motions, as the incumbent weight acts equally in all directions, it neutralizes itself, by aiding just as much as it obstructs their efforts to move and turn themselves. The case is precisely similar with respect to land animals. The vessels of the animal, together with the bones, are filled with air or some other fluid capable of supporting any weight, and whose elasticity being equal to that from without, proves an exact counterbalance to it.

467. In the barometer there is an equilibrium between the pressure of the mercury and that of the atmosphere. Now we have seen that when two fluids thus counterbalance each other, the altitudes must be inversely as the specific gravities. Accord- 411.
ingly as the specific gravity of mercury is to that of air at the surface of the earth as 13,57 to 0,00122, we shall have

$$0{,}00122 : 13{,}57 :: 30 : \frac{13{,}57 \times 30}{0{,}00122} = 333688.$$

We infer, therefore, that the height of the atmosphere, on the supposition of a uniform density throughout, is 333688 inches, or a little more than 5 miles. But the air being eminently elastic, the lower strata are compressed by the incumbent weight of those above, so that the density becomes less and less continually as we ascend. Let the weight of the column of mercury which measures the pressure of the atmosphere, exerted upon a unit of surface, be denoted by $g \, \Delta \, h$, g being the force of gravity, Δ the density of the mercury, and h the perpendicular height of the column above the level of the surface in the basin, and let the weight of the atmosphere upon the same surface be denoted by w, we shall have

$$g \, \Delta \, h = w.$$

As we ascend into the atmosphere, the weight w and the height h diminish continually, and these diminutions depend upon the elevation attained, and the law according to which the densities of the atmospheric strata decrease. If this law were known, it might be made use of for the purpose of determining the difference in the altitudes of two points above a common level, as the sea, or any assumed level. But in order to discover this law, it

is necessary to recur to certain experiments relating to the density of the air under different pressures and at different temperatures.

Fig. 130. 468. Take a recurved glass tube ABC, open at the extremity A and closed at the other extremity C; pour into it a quantity of mercury just sufficient to fill the bended part up to the horizontal line DE, so that the air confined in the shorter branch CE may be neither more nor less pressed than that contained in the longer branch AD, which communicates with the atmosphere. The mercury being at the same height, therefore, in each branch, and the communication with the external air being cut off, if we introduce, by means of a fine tunnel, more mercury, we shall observe this liquid to stand higher in the branch BA than in the other, whereby the air in EC will be condensed, the compressing force being equal to the difference of the two columns. If the space EC, supposed, for example, to be 4 inches, were reduced one half or to F, by the pressure of a column of mercury extending to H, drawing the horizontal line FG, we should find the difference GH of the two columns exactly equal to the height of the barometer at the time of the observation; so that the air contained in the space CF would be pressed by the weight of the atmosphere incumbent upon H and by the weight of another atmosphere represented by the column GH. A double pressure, therefore, reduces the bulk one half. If we continue to add to the weight by pouring in more mercury till the confined air is condensed to F' or to one third of the original space, we shall find the additional quantity necessary to this effect the same as before, that is, the column GH' will be equivalent to two atmospheres. Thus a triple pressure reduces the bulk of the confined air to one third the space. We might continue to increase the weight, and we should in every instance obtain results agreeable to the same general law.

So, on the other hand, by diminishing the natural pressure exerted upon any portion of air, we shall still find the bulk inversely proportional to the pressure. Let the tube ABC be supposed to have a bore not exceeding one tenth of an inch. A drop of mercury being introduced at the bend A, if the whole apparatus be placed under the receiver of an air-pump, and the

air be exhausted from the longer branch, till the pressure is reduced successively one half, two thirds, &c., the portion of air confined by the drop of mercury will expand, driving the drop before it, and will occupy successively, double, triple, &c., of its original bulk. We infer, therefore, universally, that *the space occupied by any given portion of air is reciprocally proportional to the pressure.*

In order that this law may hold true, however, in the strictest sense, it is to be remarked that *the air must be perfectly dry;* for the small quantity of aqueous vapor, which is ordinarily found mixed with the atmosphere, is not condensed by pressure according to the same law, as will be shown hereafter.

The instrument represented by figure 230, is called a *manometer.* It is used for the purpose of measuring the elastic force of other gases besides the atmosphere; and they are all found to be condensed and expanded according to the above law. This important property was discovered by Mariotte, and is frequently referred to under the name of the law of Mariotte.

469. Recurring to the first experiment above described, the pressure exerted upon the portion EC of confined air, when the recurved part of the tube is just filled with mercury, is that of the atmosphere, or $g \,\Delta\, h$. But this pressure is resisted and counterbalanced by the elasticity of the confined air, which by supposition is of the same density with that immediately surrounding the apparatus. We may take $g \,\Delta\, h$, therefore, as the measure of the elastic force of the air in question. This force remains the same so long as the air continues of the same density and the same temperature. If a manometer be removed from one place to another, care being taken not to change the state of the confined air, the product $g \,\Delta\, h$ which represents the elastic force does not undergo any change. But if the gravity g varies as we remove from one place to another, the height h of the mercury will also vary in the inverse proportion to g, the density Δ of the fluid being supposed to remain the same. It will hence be perceived that the variations in the heights of the mercury in the manometer are capable of rendering sensible the variations of gravity, and may even be employed in determining the augmentations or diminutions of this force arising from changes of distance with respect to the centre of the earth.

470. Let us now suppose that, the weight of the atmosphere remaining the same, the temperature of the confined air is raised; as this air expands, its bulk will be increased, and its density diminished. Now we know by the careful experiments of M. Gay Lussac and others, (1.) That all the gases dilate uniformly, at least from 32° to 212°, or from the freezing to the boiling point of water. (2.) That the dilatation arising from the same increase of heat is precisely the same for all the gases, vapors, and mixtures of gases and vapors. (3.) That the bulk of confined gas, at the temperature of 32°, being considered as unity, this common dilatation is 0,375, (or a little more than one third,) for 180°, the difference between the boiling and freezing points of water; which gives $\frac{0,375}{180} = \frac{1}{480}$ or 0,00208 for the augmentation of bulk answering to 1° of Fahrenheit. Accordingly, we shall have for the bulk or space occupied by the portion of air in question $1 + 0,00208\, n$ at the temperature denoted by n, the number of degrees above or below 32, the latter being considered as negative. This bulk or volume may be reduced to its original limits, by bringing the temperature back to 32°, or by increasing or diminishing the weight which compresses it, without altering the temperature. It would only be necessary, in this latter case, to add to or take from the weight w a portion equal to $w(0,00208)\, n$, that is, to substitute for w the weight

$$w\,(1 + 0,00208\, n),$$

which is the measure of the elastic force of the confined air reduced to its original density. Hence, *the bulk and density remaining the same, the elastic force varies with the temperature, and in the same ratio.*

If the elastic force is proportional to the density when the temperature is the same, and varies with the temperature when the density is the same, it will be easy to deduce the value of this force in terms of the two elements, on the supposition that they both vary together. Thus, putting Δ for the density of the air in question, and n for the number of degrees which marks the temperature, and p for its elastic force or pressure exerted upon the unit of surface, a being the ratio of the elastic force to the density at the temperature of 32°, we shall have

$$p = a \triangle (1 + 0{,}00208\, n). \qquad (\text{I.})$$

The coefficient a is constant for the same elastic fluid, but is different in different fluids, and requires to be determined in each particular case.

471. In applying the results above stated to the mass of air which composes the atmosphere, we take into consideration only a single vertical column of air, supposed to rest upon the surface of the earth and to extend indefinitely upward. We may conceive of the surrounding mass or atmosphere as congealed or rendered solid. If it were previously in a state of equilibrium, this state will not be disturbed by such a supposition; so that the column in question will still be in equilibrium as before. Now the force which acts upon the particles of air is gravity, which may, without sensible error, be regarded as exerting itself in the direction of the aerial column throughout its whole extent, or at least as far as it is necessary to take any account of it. Accordingly, it is necessary, in order to an equilibrium, that the density, the pressure, and the temperature should be considered as uniform throughout a horizontal stratum of infinitely small thickness. The column being composed of an infinite number of these strata or lamina, let h be the height or distance from the surface of the earth of one of these strata, $\triangle$ the density of this stratum, T its temperature, g' its gravity, p its elastic force, σ its base, and $d\,h$ its thickness. We shall have $\sigma\, p$ for the pressure exerted upon the inferior base, and $\sigma\,(p - d\,p)$ for the pressure upon the superior base; the difference $-\,\sigma\, d\,p$ must be equal to the weight $\sigma \triangle g'\, d\,h$ of this stratum. Hence, by suppressing the common factor σ, we have the equation

$$-\,d\,p = \triangle\, g'\, d\,h,$$

or, substituting for $\triangle$ its value $\dfrac{p}{a\,(1 + e\,n)}$ deduced from equation (I.), the fraction 0,00208 being for the sake of brevity represented by e,

$$-\,d\,p = \frac{p}{a\,(1 + e\,n)}\, g'\, d\,h.$$

Whence

$$\frac{d\,p}{p} = \frac{-\,g'\,d\,h}{a\,(1 + e\,n)}.$$

Nothing can be inferred from this equation until the value of n is given in terms of h. Now we know that the temperature decreases as we ascend from the surface of the earth, but the law of this decrease has not been determined in a manner altogether satisfactory. Fortunately, this law has little influence upon our results in the calculation of heights by the barometer, on account of the smallness of the coefficient e; and we may, in questions of this kind, consider the temperature as constant, provided we take for n, in each particular case, the mean of the temperatures observed at the two extreme points of the height h to be determined. Moreover, R being the radius of the earth, and g the gravity at the surface, we have, at the distance $\text{R} + h$ from the centre,

$$g' = \frac{g\,\text{R}^2}{(\text{R} + h)^2},$$

since this force varies in the inverse ratio of the square of the distance. The preceding equation becomes, by this substitution,

$$\frac{d\,p}{p} = \frac{-\,g\,\text{R}^2\,d\,h}{a\,(1 + e\,n)\,(\text{R} + h)^2}.$$

Whence, by integrating on the supposition that n is constant, we have

$$\log.\,p = \frac{m\,g\,\text{R}^2}{a\,(1 + e\,n)} \cdot \frac{1}{\text{R} + h} + C;$$

Cal. 112. m being equal to 0,434295, *log.* denotes the common logarithm of p. To determine the constant C, let ϖ be the value of p answering to $h = 0$; and we shall have

$$\log.\,\varpi = \frac{m\,g\,\text{R}}{a\,(1 + e\,n)} + C.$$

Consequently, by subtracting the preceding equation from this, we obtain

$$\log.\,\frac{\varpi}{p} = \frac{m\,g\,\text{R}}{a\,(1 + e\,n)} \cdot \frac{h}{\text{R} + h}. \qquad \text{(II.)}$$

This equation, taken in connection with equation (I.), gives the values of p and Δ in terms of h. Thus we have equations con-

taining the laws of the density and elastic force of the air which belong to a state of equilibrium in the atmosphere.

472. To make use of equation (II.) for the purpose of measuring heights by means of the barometer, let us suppose the barometric altitude at the surface of the earth and at the height h to be known by actual observation, and let them be denoted respectively by w, w', the corresponding temperatures of the mercurial columns being represented by τ, τ'. The expansion of mercury being $\frac{1}{9742}$ or 0,0001025, that is, 0,0001 nearly, for each degree of Fahrenheit's scale, if D be the density corresponding to the temperature τ of the mercury at the first station,

$$\mathrm{D}\,(1 + 0{,}0001)\,(\tau - \tau')$$

will be the density which answers to the temperature of the mercury at the second station. Accordingly we have

$$\varpi = \mathrm{D}\,g\,w, \quad \text{and} \quad p = \mathrm{D}\,g'\,w'\,(1 + 0{,}0001)\,(\tau - \tau').$$

The correction for the upper barometric column on account of difference of temperature being made agreeably to this formula, we may consider w' as representing the length of this column thus corrected. Whence, dividing the first of the above equations by the second, we have

$$\frac{\varpi}{p} = \frac{g\,w}{g'\,w'} = \frac{w}{w'} \times \frac{(\mathrm{R} + h)^2}{\mathrm{R}^2}, \quad (\text{III.})$$

substituting for g' its value $\frac{g\,\mathrm{R}^2}{(\mathrm{R} + h)^2}$; and consequently,

$$\log.\,\frac{\varpi}{p} = \log.\,\frac{w}{w'} + 2\log.\,\left(1 + \frac{h}{\mathrm{R}}\right), \qquad (\text{III.})$$

since

$$\frac{(\mathrm{R} + h)^2}{\mathrm{R}^2} = \left(\frac{\mathrm{R} + h}{\mathrm{R}}\right)^2 = \left(1 + \frac{h}{\mathrm{R}}\right)^2.$$

Let T, T′, be the temperatures respectively of the air at the surface of the earth and at the height h; T, T′, will generally differ from τ, τ', the temperatures of the mercury in the barometer, since the latter is not ordinarily allowed sufficient time to acquire the temperature of the surrounding air. T, T′, are to

be taken by means of a thermometer suspended in the air, while τ, τ', are supposed to be indicated by a thermometer attached to the barometer. We take $n = \frac{\text{T} + \text{T}'}{2} - 32$. Moreover, the coefficient $\frac{1}{480}$ or 0,00208, representing the elastic force, requires to be increased somewhat for the purpose of taking account, as far as can be done, of the quantity of water in a state of vapor which is at all times mixed with the air in a greater or less quantity. Indeed, under the ordinary pressure of the atmosphere, the density of aqueous vapor is to that of air, as 10 to 14; consequently, the atmosphere is so much the lighter according as it is composed in a greater degree of this vapor. Now it contains so much the more vapor according as its temperature is more raised, whereby, when the air is dilated by heat, its weight must be diminished in a higher ratio than that of its augmentation of bulk. We increase the coefficient 0,00208 therefore to 0,00223 or $\frac{1}{449}$, which has been found by actual trial to give the most correct results. We have, accordingly,

$$e\,n = 0{,}00223\left(\frac{\text{T} + \text{T}'}{2} - 32^\circ\right).$$

We now substitute in equation (II.) for $e\,n$ the above value, and for $\log. \frac{\varpi}{p}$ the value found in equation (III.), and we shall obtain

$$\log. \frac{w}{w'} + 2 \log. \left(1 + \frac{h}{\text{R}}\right) = \frac{m\,g\,\text{R}}{a\left(1 + 0{,}00223\left(\frac{\text{T} + \text{T}'}{2} - 32^\circ\right)\right)} \times \frac{h}{\text{R} + h}.$$

Whence

$$h = \log. \frac{w}{w'} + 2 \log. \left(1 + \frac{h}{\text{R}}\right) \frac{a\left(1 + 0{,}00223\left(\frac{\text{T} + \text{T}'}{2} - 32^\circ\right)\right)(\text{R} + h)}{m\,g\,\text{R}}$$

$$= \frac{a}{m\,g}\left(1 + 0{,}00223\left(\frac{\text{T} + \text{T}'}{2} - 32^\circ\right)\right) \log. \frac{w}{w'} + 2 \log. \left(1 + \frac{h}{\text{R}}\right)\left(1 + \frac{h}{\text{R}}\right) \quad \text{(IV.)}$$

The best means of determining the coefficient $\frac{a}{m\,g}$ of this

formula, is to make use of a height (or rather a number of heights), well known by actual measurement, or by trigonometrical operations. We then substitute for h this known value, and for w, w', T, T′, the lengths of the barometrical columns, and the temperature of the air at the two stations respectively, and for R the mean radius of the earth, namely 3481280 fathoms. We shall thus have an equation from which the value of $\frac{a}{m\,g}$ is readily deduced once for all. Taking the mean result of a great number of observations conducted with the greatest care, by M. Ramond, we find $\frac{a}{m\,g}$, for the latitude 45°, * equal to 18336 † metres, or 10026 English fathoms. This is on the supposition of a temperature of 32°, and agreeably to what has been said, it may be increased or diminished by adding or subtracting $\frac{1}{449}$ or a 0,00223 part for each degree above or below 32°. We can therefore reduce it to 10000, instead of 10026, by supposing the temperature somewhat lower. Thus, since 26 is 0,0026 of 10000

$$0{,}00223 : 0{,}0026 :: 1^\circ : 1^\circ{,}16.$$

If, therefore, we subtract 1°,16 from 32°, we shall have 30°,84 or 31° nearly, for the temperature at which the constant coefficient is 10000 fathoms.

473. Since this coefficient contains g, it must vary with g, that is, with the latitude. Now, according to the law of the va-

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* This coefficient was actually determined for the latitude of about 43°. But the correction for small distances in latitude is so inconsiderable, that it may be regarded as nothing. Moreover, the coefficient, if corrected at all, would require to be diminished, and it is thought on the whole less liable to error by excess than by deficiency.

† The coefficient deduced theoretically from the relative densities of mercury and air, as determined by Biot and Arago, allowance being made for humidity, is 18334,1 metres, differing less than 2 metres, that is, less than 2 × 39,371 inches, from the above.

riation of gravity in different latitudes, if g represent the value of this force in the latitude of 45°, and g' that of any other latitude L, we shall have

$$g' = g\,(1 - 0{,}002837 \cos 2\,L). *$$

At 45°, therefore, where $\cos 2\,L = 0$, $g' = g$, or the correction, is 0; and for higher latitudes the correction is —, or subtractive, and for lower latitudes it is + or additive. Whence, generally,

$$\frac{a}{m\,g} = 10000^{\text{fath.}}\,(1 + 0{,}002837 \cos.\, 2\,L.)$$

By means of this value of $\frac{a}{m\,g}$, substituted in equation (IV.)

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* The value of g' in different latitudes depends upon the particular figure of the terrestrial spheroid, the determination of which belongs to astronomy. We will merely observe in this place, that a comparison of articles 346, 347, conducts us directly to the equation

$$\frac{g'}{g} = \frac{a'}{a},$$

a, a', being the lengths of the pendulum corresponding to the parts of the earth in which the intensities of gravity are g, g', respectively. Now it is found that the general expression for the length of the seconds pendulum, the day being divided into 100000 seconds, is

$$a' = \overset{\text{metre.}}{0{,}739502} + \overset{\text{metre.}}{0{,}004208}\,(\sin L)^2.$$

Hence, since

$$(\sin L)^2 = \tfrac{1}{2}\,(1 - \cos 2\,L), \text{ and } (\sin 45°)^2 = \tfrac{1}{2},$$

Trig. 20. Trig. 27. we shall have

$$\frac{a}{a'} = \frac{g}{g'} = \frac{0{,}739502 + 0{,}002104}{0{,}739502 + 0{,}004208\,(\sin L)^2},$$

$$= \frac{1}{1 - 0{,}002837 \cos 2\,L};$$

therefore,

$$g' = g\,(1 - 0{,}002837 \cos 2\,L).$$

In the original memoir of M. Ramond, the coefficient stood 0,002845, and it was thus copied by Laplace and others. It was afterward corrected by M. Oltmanus, and the error acknowledged by the author in a separate edition of the memoir.

we shall be able to determine the height h in any part of the earth, when w, w', T, T', corresponding to the extreme points of h, are known; or we may retain the coefficient 10000 unaltered, and apply the correction to the result, according to the following table;

Latitude.	Correction.
0°	$+\frac{1}{352}$ of the approximate height.
5°	$+\frac{1}{358}$
10°	$+\frac{1}{375}$
15°	$+\frac{1}{407}$
20°	$+\frac{1}{460}$
25°	$+\frac{1}{548}$
30°	$+\frac{1}{705}$
35°	$+\frac{1}{1030}$
40°	$+\frac{1}{2030}$
45°	$+ 0$
50°	$-\frac{1}{2030}$
55°	$-\frac{1}{1030}$
60°	$-\frac{1}{705}$
65°	$-\frac{1}{548}$
70°	$-\frac{1}{460}$
75°	$-\frac{1}{407}$
80°	$-\frac{1}{375}$
85°	$-\frac{1}{358}$
90°	$-\frac{1}{352}$

474. As the fraction $\frac{h}{\text{R}}$ is always very small, we shall have very nearly the value of h, independently of the term containing this fraction; by substituting the approximate value thus obtained for h, in the fraction $\frac{h}{\text{R}}$, we shall have very nearly the correction due to the variation of gravity at different elevations in the same latitude; and by substituting the value of h thus corrected in the fraction $\frac{h}{\text{R}}$ we can approximate the true height still more nearly. But this second substitution is altogether superfluous in the cases which ordinarily occur. Indeed, except where h is very great, we may neglect $\frac{h}{\text{R}}$ entirely, and the general formula then becomes

$$h = \frac{a}{m\,g}\left(1 + 0{,}00223\ \left(\frac{\text{T} + \text{T}'}{2}\right) - 32^\circ\right) \log. \frac{w}{w'}.$$

It will be perceived that $\frac{a}{m\,g}$ may require some modification, in order that the formula in this state should adapt itself to observed heights or known values of h; and indeed the observations of M. Ramond give, for the value of the coefficient to be employed in this formula, $\frac{a}{m\,g} = 18393$ metres or 10031 fathoms, exceeding the former by 5 fathoms. Accordingly the depression of the temperature below 32°, required in order to change this to the more convenient form of 10000, will be found to be 1°,45; retaining the coefficient 10000, therefore, we have only to suppose the temperature 32° — 1°,45 or 30°,55. As this differs less than half a degree from 31°, and as we can seldom be certain of the temperature of the air to a greater degree of accuracy, we may still use the same formula without any other change than the omission of the term depending on $\frac{h}{\text{R}}$. We have hence a very simple, convenient, and, for common cases, sufficiently exact formula, namely,

$$h = 10000\ (1 + 0{,}00223\left(\frac{\text{T} + \text{T}'}{2}\right) - 31^\circ)\ (\log. w - \log. w').$$

This being adapted to the latitude of 45°, when the barometrical observations relate to a place on a parallel considerably distant either north or south, it will be seen directly by the foregoing table when it is necessary to apply a correction for difference of latitude, and what this correction is. It will be recollected that the lengths of the barometric columns w, w', which represent the weights of the atmosphere respectively at the two stations, are supposed to be reduced to the same temperature. The upper column w' is usually the coldest, and consequently too short. Now, according to the rate of expansion or contraction already mentioned, as 1 inch is shortened 0,0001 for each degree, a column of 25 inches will be shortened 0,0025 of an inch for each degree of depression, and consequently 0,01 for every — 4°; and each portion of 2,5 inches will be shortened one tenth part of this, or 0,001 for the same amount of depres-

sion or $-4°$. In common chamber barometers, the lengths of the columns are read off to a 0,01 of an inch, and in the best to the $\frac{1}{500}$ or 0,001 of an inch.

The best time for taking observations with the barometer for the purpose of calculating heights is during settled weather and at mid-day. Observations taken in the morning or evening are much more liable to be erroneous on account of ascending and descending currents of air which take place at these times. Moreover a course of continued observations is more likely to lead to accurate results than single observations. In the calculation of very small heights near the level of the ocean, it is common where great accuracy is not required, to dispense with the formula and adopt the following rule, namely, *as* 0,1 *inch is to the difference in the barometric columns, so is* 87 *feet to the approximate difference of level required;* which is to be corrected, if necessary, for the difference from 31° of the mean temperature of the air at the two stations.

Thus, under a pressure of 30 inches of mercury at the temperature of 50°, 0,1 of an inch of mercury answers to 87 feet of atmosphere. It will be seen moreover, that, as 0,1 of an inch of mercury is equivalent to 87 feet of air, 0,01 of an inch answers to 8,7 feet; 0,001 of an inch to 0,87 feet; and $\frac{1}{500}$ of an inch to 1,74. Hence in a good mountain barometer, graduated to 500dths of an inch, there will be a sensible difference in the pressure of the air arising from a change of altitude of less than two feet, or two thirds the length of the instrument.

Formula (IV.) is essentially the same with that given by Laplace in the 10th book of the *Mécanique Céleste*, but simplified after the example of Poisson, and reduced to English measures. The following example will serve to illustrate every part of this formula.

At the lower of two stations, the mercury in the barometer was observed to be 29,4 inches, and its temperature 50°, that of the air being 45°; and at the upper station, the height of the barometer was 25,19, its temperature 46°, and that of the air 39°, the latitude of the place being 30°.

In this case, we have

$$\tau - \tau' = 50° - 46° = 4°; \frac{T + T}{2} - 31° = \frac{45° + 39°}{2} - 31° = 11°;$$

and

$$\cos 2\,L = \cos 2 \times 30^\circ = \cos 60^\circ = \tfrac{1}{2}.$$

Whence

$w = 29{,}4$		log. . .	1,46835
$w' = 25{,}2003$		log. . .	1,40141
	669,4	log.. .	2,82569
11°		log.. .	1,04139
0,00223 . . .		log.. .	$\bar{3}$,34830
1st correction	16,42	log.. .	1,21538
	685,82	log.. .	2,83621
$\frac{1}{2}$ 0,002837 . . .		log.. .	$\bar{3}$,15168
2d correction	0,97	log.. .	$\bar{1}$,98789
	686,79	log.. .	2,83682
R = 3481280 . .		log.. .	6,54174
$\frac{h}{\text{R}} = 0{,}0002$		log.. .	$\bar{4}$,29508
$\left(1+\frac{h}{\text{R}}\right)\left(1+\frac{h}{\text{R}}\right) = 1{,}0004$ . .		2 log...	0,00034
3d correction	3,4		
	690,19	fathoms.	

Laplace's formula, applied to the same example, gives 688,97 fathoms, differing from the above only 1,22 fathoms; whereas, by Sir George Shuckburgh's method, in which no account is taken of the variation of gravity, either for difference of latitude or difference of elevation in the same latitude, the result is 685,125. This corresponds with the approximate height derived from the first correction in the above example.

475. We have already mentioned, that, unless very particular precautions are taken, mercury is depressed in glass tubes, and that this depression is inversely proportional to the diameter of the tube. It is always indicated, moreover, when it takes place by the upper surface being convex. It is not necessary

to have regard to this circumstance in the calculation of heights by the barometer, where the two observations are taken with the same instrument, since the difference in the length of the barometric columns would be the same, whether they were corrected or not.* But in order that observations by different instruments, liable to different capillary effects, may be strictly compared with each other, a correction should be applied, which may be readily done by means of the following table.

Interior diameter of the tube in English inches.	Depression of the Mercury.
0,6	0,005
0,5	0,007
0,4	0,015
0,35	0,025
0,3	0,036
0,25	0,050
0,2	0,067
0,15	0,092
0,1	0,140

* Also in a syphon barometer, or one in which the tube, instead of entering a basin, turns up at the bottom and continues of the same bore, as in figures 230, 232, since the capillary effect is the same in both branches, the observed altitude reckoned from the surface in the shorter branch, would not be affected by the correction.

HYDRODYNAMICS.

Of the Discharge of Fluids through Apertures in the Bottom and Sides of Vessels.

476. If a fluid be made to pass through a canal or tube of variable bore, kept constantly full, and the velocity be the same in every part of the same section, since for any given time the same quantity of fluid must pass through every section, this quantity must be equal to the area of the section multiplied by the velocity. σ, σ', being the areas of two sections, and v, v', the velocities at these sections, we shall have

$$\sigma\, v = \sigma'\, v',$$

and hence

$$\sigma : \sigma' :: v' : v,$$

that is, *the velocities in different sections are inversely as the areas of the sections.*

The case here supposed is purely theoretical, and can never occur in practice, since on account of friction, the velocity is always greatest at the surface in a canal, and at the axis in a tube.

477. Let *MNOP* represent a vessel filled with a fluid up to Fig. 233.
GH, *CD* an aperture, very small compared with the bottom *MP*, *CIKD* the column of fluid directly above the aperture, and *CABD* the lowest lamina or stratum of this fluid, immediately contiguous to the aperture. Also let v denote the velocity acquired by a heavy body in falling freely through *BD*, the height of the stratum, and u the velocity which the same stratum would

acquire in falling through the same space by the pressure of the column $CIKD$. If we suppose the lowest stratum $ACDB$, to fall as a heavy body through the height BD, the moving force will be its own weight. But if we suppose it to be urged by its own weight, together with the pressure of the incumbent column of fluid $CIKD$, through the same space, the velocity in the former case will be to that in the latter, as the moving forces and the times in which they act, the mass moved being the same in both cases.
27. But the moving forces are to each other, as the heights BD, KD,
264. and the times in which they act, the space being the same, are inversely as the velocities. Accordingly,

$$v : u :: \frac{BD}{v} : \frac{KD}{u}.$$

Whence

$$v^2 : u^2 :: BD : KD,$$

or

$$v : u :: \sqrt{BD} : \sqrt{KD}.$$

Now v is the velocity which a heavy body would actually acquire in falling through the space BD, and as the velocities, other things being the same, are as the square roots of the spaces,
268. u the velocity of the issuing fluid is that which a heavy body would acquire in falling through KD, the height of the fluid above the orifice. Therefore, *the velocity with which a fluid is discharged from the bottom of a vessel is equal to that acquired by a heavy body in falling through a space equal to the height of the fluid above the orifice.* Also if a pipe $A'B'C'D'$ be inserted horizontally, or inclined in any way to the horizon, it may be shown, in like manner, since the pressure of fluids is equal in all directions, that the fluid will be discharged with the same velocity as before. It will accordingly ascend to the level of the fluid in the vessel, all obstructions being removed; and it is found in fact, under the most favorable circumstances, nearly to reach this point. It follows, moreover, from what is above laid down, that if apertures be made at different distances s, s', s'', below the surface, the velocities at these points, and consequently the quantities of fluid discharged at these points, from apertures of

the same size, in the same time, the vessel being kept filled to the same level, will be as $\sqrt{s}$, $\sqrt{s'}$, $\sqrt{s''}$. The actual velocity at the distance s below the surface of the fluid in the vessel will be $\sqrt{2gs}$, and the quantity Q discharged in the time t, through an aperture whose area is σ, is as follows, namely, 277.

$$Q = \sigma t \sqrt{2gs}.$$

478. What is above said of the velocity of a fluid discharged from jets or apertures, is true only of the middle filament of particles issuing through the centre of the aperture, which are supposed to experience no retardation, and which, in fact, suffer no other retardation than what arises from the resistance of the air, and their mutual adhesion and attrition against each other. But those which issue near the edges of the aperture suffer a much greater resistance, and are accordingly much more retarded. Hence it follows that the mean velocity of the whole column of discharged fluid will be considerably less than that indicated by the above theory.

479. Sir Isaac Newton discovered a contraction in the vein of discharged fluid, and found that at a distance from the orifice of about a diameter of this orifice, the section of the vein or stream was diminished nearly in the ratio of $\sqrt{2}$ to $\sqrt{1}$. Hence he concluded that the velocity of the fluid after passing the aperture was increased in this proportion, the same quantity passing through a narrower space in the same time.

According to some very accurate experiments of Bossut, the actual discharge through a hole made in the side or bottom of the vessel, is to the theoretical as 1 to 0,62, or nearly as 8 to 5. The theoretical discharge must, therefore, be diminished in this ratio to obtain the actual discharge.

If the water issues, not through an aperture in the side or bottom of the vessel, but through a pipe from 1 to 2 inches in length, inserted in the aperture, the contraction of the vein is prevented, and the actual discharge becomes to the theoretical, as 8 to 10, or as 4 to 5. In this way, therefore, the discharge is increased nearly in the ratio of 4 to 3.

The theoretical discharge, the discharge through an additional tube, and that through a simple perforation in the side, are as the numbers 16, 13, and 10 nearly.

480. When an upright cylinder or prismatic vessel is suffered gradually to discharge itself, the velocity of the descending surface of the fluid is to the velocity at the orifice, as the area of the latter is to that of the former, and this is a constant ratio; consequently the velocity of the descending surface varies as the velocity at the orifice, or as $\sqrt{s}$; that is, the velocity of the descending surface varies as the square root of the space to be described by it; so that this corresponds exactly to the case of a body projected perpendicularly upward; whence, as the re-
270. tarding force is constant in the instance just referred to, it must be constant also in the case before us. Therefore, *when a vessel of the above description is suffered to discharge itself, the velocity of the descending surface and that of the discharged fluid will be uniformly retarded.*

Suppose a body, urged by a constant force, as that of gravity, to describe a space, as 1 rod, for instance, in the first second;
267. the spaces being as the squares of the times, it will describe 4 rods in two seconds, 9 rods in 3 seconds, and so on; and the spaces described in the first second, second second, &c., will evidently be the differences of these, namely, 1 — 0, 4 — 1, 9 — 4, 16 — 9, &c., that is, the series of odd numbers, 1, 3, 5, 7, 9, &c. Accordingly, these numbers, taken in the inverse order, represent the spaces described in equal times by a body uniformly retarded; they represent, moreover, as will be seen from what is above proved, the quantities of fluid discharged in equal times from an aperture in the bottom of a prismatic vessel. Hence, if it were proposed to construct a *clepsydra*, or water-clock, by means of a prismatic or cylindrical vessel, having an aperture
Fig. 234. in the bottom, let the height DB of a vessel which would be completely exhausted in a given time, as 12 hours, be divided from the top downward into portions represented by the numbers 23, 21, 19, &c., down to 1, which will require the height DB to be divided into 144 equal parts, and these portions 23, 21, &c., will be the spaces through which the upper surface will descend in each successive hour of the exhaustion.

481. If x denote the space through which the upper surface A descends in the time t, the velocity of the discharged fluid, represented by $\sqrt{2g(s-x)}$, will vary continually, but may be considered as constant during the indefinitely small time dt; so that in this time there will escape through the orifice, a prism of the fluid having the area σ of this orifice for its base, and $\sqrt{2g(s-x)}$ for its altitude. Thus the quantity discharged during the instant dt is $\sigma\, dt\sqrt{2g(s-x)}$. But during the same time the upper surface has descended through the space dx, and the vessel has lost a prism or cylinder of the fluid, whose base is A, and altitude dx, and whose bulk or volume is A dx, whence

$$\text{A}\, dx = \sigma\, dt\sqrt{2g(s-x)},$$

and

$$dt = \frac{\text{A}\, dx}{\sigma\sqrt{2g(s-x)}}. \qquad (\text{I.})$$

As the area A will be given in functions of x, by the form of the vessel, the second member of this equation may be considered as containing only the variable quantity x, and it will be easy in most cases, by simply integrating, to discover the successive depressions of the surface, and the discharges of the fluid, from any vessel of a known form.

482. Let the vessel, for example, be an upright prism or cylinder; A in this case will be constant, and we shall have

$$t = \frac{\text{A}}{\sigma\sqrt{2g}} \cdot \int \frac{dx}{\sqrt{s-x}} = -\frac{2\,\text{A}}{\sigma\sqrt{2g}}\sqrt{s-x} + C.$$

Now when the time t is 0, the depression of the upper surface A is 0 also; thus we have at the same time $x = 0$, and $t = 0$; this condition determines the constant quantity C to be

$$\frac{2\,\text{A}}{\sigma\sqrt{2g}}\sqrt{s},$$

and gives for the time of depressing the upper surface through the space x, as follows,

$$t = -\frac{2\,A}{\sigma\sqrt{2g}}\sqrt{s-x} + \frac{2\,A}{\sigma\sqrt{2g}}\sqrt{s} = \frac{2\,A}{\sigma\sqrt{2g}}(\sqrt{s}-\sqrt{s-x}).$$

To find the time of completely emptying the vessel, we have only to make $x = s$, in which case the preceding expression will become

$$t = \frac{2\,A}{\sigma\sqrt{2g}}\sqrt{s} = \frac{A}{\sigma\sqrt{2g}}\sqrt{4s} = \frac{A}{\sigma}\sqrt{\frac{2s}{g}}.$$

But from what is above shown, we have Q or $A\,s = \sigma\,t\sqrt{2gs}$, from which we obtain

$$t = \frac{A\,s}{\sigma\sqrt{2gs}} = \frac{A}{\sigma}\sqrt{\frac{s}{2g}} = \frac{A}{2\,\sigma}\sqrt{\frac{2s}{g}}.$$

By comparing this result with the preceding, it will be seen that *when a vessel is suffered to exhaust itself, the time employed is just double that required to discharge the same quantity when the vessel is kept full.* The same conclusion might indeed be drawn from articles 266, 270.

483. Let the vessel be any solid generated by the revolution of a curve. The axis being vertical, A will be the area of a circle which has for its radius the ordinate y of the generating curve, that is, if $\pi = 3{,}14159$ &c., $A = \pi\,y^2$. Substituting this value for A in
Geom. 290. the equation (I.) of article 481, we have

$$t = \frac{\pi}{\sigma\sqrt{2g}}\cdot\int\frac{d\,x\,y^2}{\sqrt{s-x}}.$$

In any particular examples, it will be necessary to put for y its value deduced, in terms of x, from the equation of the generating curve.

484. Let *ABCD* be the vertical side of a vessel, *EFGH* a
Fig 235. rectangular notch in it, and let *IL i l* be a rectangular parallelogram whose breadth *I i* is infinitely small compared with *EG*. The velocity with which the fluid would escape at *GH*, is to the velocity with which it would escape from *IL i l*, as $\sqrt{EG}$ to $\sqrt{EI}$,

and the quantities of fluid discharged in a given time through 268.
indefinitely small parallelograms at these depths are in the same ratio. But the parabolic curve EKH being drawn, having EG for its axis, we have

$$EG : EI :: \overline{GH}^2 : \overline{IK}^2;$$

and consequently, Trig.176.

$$\sqrt{EG} : \sqrt{EI} :: GH : IK;$$

whence the quantities discharged through indefinitely small parallelograms at the depths EG, EI, are to each other as the ordinates GH, IK, and the sum of all the quantities discharged through all the parallelograms of which the rectangle $EFGH$ is composed, is to the sum of all the quantities discharged through as many parallelograms at the depth EG, as the sum of all the elements $IKki$ of the parabola, to the sum of all the corresponding elements $ILli$ of the rectangle; that is, as the area of the parabola $EKHG$ to the area of the rectangle $EFGH$; in other words, the quantity running through the notch $EFGH$ is to the quantity running through an equal horizontal area at the depth EG, as $EKHG$ to $EGHF$, that is, as 2 to 3. Therefore the Cal. 94.
mean velocity of the fluid in the notch is equal to two thirds of that at the greatest depth GH.

485. If a small aperture be made in the side of a vessel kept Fig.236.
filled to the same height, the fluid will spout out horizontally with the velocity acquired by a heavy body in falling freely through the height of the fluid above the aperture, and this velocity combin- 477.
ed with the perpendicular velocity arising from the action of gravity, will cause each particle, and consequently the whole jet to describe a parabola. Now the velocity with which the fluid 303.
is expelled from any aperture, as G, is such as would, if uniformly preserved, carry a particle through a space equal to $2\,BG$ 266.
in the time of its natural descent through BG; accordingly, if the direction of the aperture be horizontal, the action of gravity being at right angles to it will cause the particle to descend through the height GD in the same time that would be required in case of a natural descent through GD, if no other force were

267. exerted upon the particle. Hence the squares of the times being as the spaces, or the times simply as the square roots of the spaces, $\sqrt{BG}$ is to $\sqrt{GD}$ as the time employed in describing BG to the time required to reach the horizontal plane DF. But in the time employed in describing BG, the particle would be carried uniformly and horizontally by the velocity thus acquired, through a space equal to $2\,BG$; therefore, to find the amplitude or horizontal range DE of the jet, we have the proportion

Geom. 215.

$$\sqrt{BG} : \sqrt{GD} :: 2\,BG : DE = \frac{2\,BG\,\sqrt{GD}}{\sqrt{BG}}$$

$$= 2\,\sqrt{BG \cdot GD} = 2\,GH.$$

As the same reasoning may be used with respect to any other point in BD, *if upon the height of the fluid* BD *as a diameter we describe a semicircle* BKD, *the horizontal distance to which the fluid will spout from any point will be twice the ordinate of the circle drawn through this point, the distance being measured on the plane of the bottom of the vessel.*

486. It will hence be perceived, that if apertures be made at equal distances G, L, from the top and bottom of the vessel, the horizontal distances DE to which the fluid will spout from these apertures will be equal; and that the point I, bisecting the altitude, is that from which the fluid will spout to the greatest distance, this distance DF being equal to twice the radius of the semicircle or to the altitude BD of the fluid.

487. If the fluid issue obliquely instead of horizontally, the curve described will still be parabolic, and the horizontal range, &c. of the jet may be calculated as in the case of other projectiles.
Fig.237. Let the aperture C be inclined, for example, upward at different angles. CB will be equal to s, the space through which a body must fall to acquire the velocity of projection, and equal to the dis-
477. tance CF, CF', of the foci of the several parabolas, traced by particles issuing with different angles of elevation. Hence BE
Trig.172. is the directrix to these parabolas, and the circle described from the centre C, and with the radius BC, will pass through the several foci F, F', &c. Let CE, for instance, be the direction of the jet, and draw CF making the angle ECF equal to BCE; let

fall the perpendicular FH, and take HG equal to HC, the distance CG will be the horizontal range of the jet. But

$$CG = 2\,CH = 2\,CF \times \cos FCH = 2\,CB \times \sin 2\,ECF.$$

Therefore, when the angle of elevation is 45°, the focus of the parabola falls on the horizontal line at F', and the range CK is then the greatest possible, being double the altitude CB.

Of the Motion of Gases.

488. To determine with what velocity the air or any other gas will rush into a void space, when urged by its own weight, we proceed according to a method analogous to that by which the motion of liquids is determined. When the moving force and the mass or matter to be moved vary in the same proportion, the velocity will continue the same, since $v = \frac{p}{m}$. 28.

Thus, if there be similar vessels of air and water, extending to the top of the atmosphere, on the supposition of a uniform density throughout, they will be discharged through equal and similar apertures with the same velocity; for in whatever proportion the quantity of matter moving through the aperture be varied by a change of density, the pressure which forces it out acting in circumstances perfectly similar will vary in the same proportion. Hence it follows that *the air rushes into a void with the velocity which a heavy body would acquire by falling from the top of the atmosphere, this fluid being supposed to be of a uniform density throughout.*

The height of a uniformly dense or *homogeneous* atmosphere being 27807 feet, according to article 467, and $g = 32{,}2$, we shall have for the velocity in question

$$v = \sqrt{2\,g\,h} = \sqrt{2 \times 32{,}2 \times 27807} = 1338.$$ 277.

489. But as the space into which the air rushes becomes more and more filled with air, the velocity must be diminished continually. Indeed whatever be the density of this rarer air, its elasticity, varying with its density, will balance a proportional part

of the pressure of the atmosphere, and it is the excess of this pressure only which constitutes the moving force, the matter to be moved being the same as before. Let D be the natural density of the atmosphere, and Δ the density of that which opposes itself to the motion in question. Let p be the pressure of the atmosphere, or the force which impels it into a void, and ϖ the force with which this rarer air would rush into a void; from the proportion

$$D : \Delta :: p : \varpi = \frac{p \Delta}{D},$$

we shall have, for the moving force sought, $p - \frac{p \Delta}{D}$. Again, let v be the velocity of air rushing into a void under the pressure p, and u the velocity of air under the same pressure rushing into rarefied air of the density Δ. Since the pressures are as the heights producing them, the fluid being supposed of a uniform density throughout, we shall have

$$v : u :: \sqrt{p} : \sqrt{p - \frac{p \Delta}{D}} :: 1 : \sqrt{1 - \frac{\Delta}{D}};$$

whence $u = v \times \sqrt{1 - \frac{\Delta}{D}}$, no allowance being made for the inertia of the rarer air, which being displaced must oppose a certain resistance.

490. Let it be proposed to determine the time t in seconds in which the air will flow into a given exhausted vessel, until the air shall have acquired in the vessel a certain density Δ.

Suppose h the height due to the velocity v, b the bulk or capacity of the vessel, and σ the area of the aperture, the measure in each case being in feet. Since the quantity of air necessary to fill the vessel will depend upon the size of the vessel, and also upon the density of the air, $b \Delta$ will represent this quantity, the differential of which is $b\, d\, \Delta$. The velocity of influx at the first instant is $v = \sqrt{2 g h}$; and when the air in the vessel has acquired the density Δ, that is, at the end of the time t, the velocity is

$$u = \sqrt{2 g h} \times \sqrt{1 - \frac{\Delta}{D}} \text{ or } \sqrt{2 g h \times \frac{D - \Delta}{D}}.$$

Hence the rate of influx, which may be measured by the infinitely small quantity of air passing the aperture during the instant $d\,t$ with this velocity, will be denoted by

$$\sqrt{2\,g\,h \times \frac{\text{D} - \Delta}{\text{D}}} \times \text{D}\,\sigma\,d\,t = \sigma\,d\,t\sqrt{2\,g\,\text{D}\,h\,(\text{D} - \Delta)}.$$

Putting these two values of the rate of influx equal to each other, we have

$$\sigma\,d\,t\sqrt{2\,g\,\text{D}\,h\,(\text{D} - \Delta)} = b\,d\,\Delta,$$

and

$$d\,t = \frac{b}{\sigma\sqrt{2\,g\,\text{D}\,h}} \times \frac{d\,\Delta}{\sqrt{\text{D} - \Delta}}.$$

Hence, by integrating, we obtain

$$t = \frac{b}{\frac{1}{2}\,\sigma\sqrt{2\,g\,\text{D}\,h}} \times \sqrt{\text{D} - \Delta} + C.$$

To find the constant C, it will be observed, that when $t = 0$, $\Delta = 0$, and $\sqrt{\text{D} - \Delta} = \sqrt{\text{D}}$. We have, therefore, for the corrected integral

$$t = \frac{b}{\sigma\sqrt{\frac{1}{2}\,g\,\text{D}\,h}} \times \left(\sqrt{\text{D}} - \sqrt{\text{D} - \Delta}\right).$$

491. When $\text{D} = \Delta$, the motion ceases, and the value of t, or the time of completely filling the vessel, becomes

$$\frac{b\sqrt{\text{D}}}{\sigma\sqrt{\frac{1}{2}\,g\,\text{D}\,h}}, \text{ or } \frac{b}{\sigma\sqrt{\frac{1}{2}\,g\,h}}, \text{ or } \frac{b}{4\,\sigma\sqrt{h}}, \text{ nearly.}$$

Suppose, for example, the capacity of the vessel to be 8 cubic feet, or nearly a wine hogshead, and that the aperture by which air of the ordinary density, or 1, enters, is an inch square, or $\frac{1}{144}$ of a foot. In this case $4\sqrt{h} = 4\sqrt{27807} = 668$, nearly; and hence

$$t = \frac{8}{\frac{1}{144}668} = \frac{1152}{668} = 1'',72 \text{ nearly.}$$

If the aperture be only $\frac{1}{100}$ of a square inch, or the side $\frac{1}{10}$, the time of completely filling the vessel will be 172″ nearly, or a little less than 3′.

If the experiment be made with an aperture cut in a thin plate,
479. we shall find the time greater nearly in the ratio of 62 or 63 to 100, as we have already remarked with respect to water flowing through small orifices.

492. We can find, in like manner, the time necessary for bringing the air in the vessel to any particular density, as $\frac{3}{4}$ of that of air in its ordinary state. For the only variable part of the integral, above found, is $\sqrt{\text{D} - \Delta}$, which in this case becomes $\sqrt{1 - \frac{3}{4}} = \frac{1}{2}$, and gives $\sqrt{\text{D}} - \sqrt{\text{D} - \Delta} = \frac{1}{2}$; hence, if the aperture were a square, each side being $\frac{1}{10}$ of an inch, the time sought would be $\frac{1}{2}$ 172″, or 86″, nearly.

493. If the air in the vessel be compressed by a weight acting on the movable cover AD, the velocity of the expelled
Fig. 238. air may be determined thus. Let the additional pressure be denoted by q, and the density thence resulting by D'; we shall then have

$$p : p + q :: \text{D} : \text{D}',$$

and

$$p : p + q - p :: \text{D} : \text{D}' - \text{D},$$

which gives

$$q = p \times \frac{\text{D}' - \text{D}}{\text{D}}.$$

Now, since the pressure which expels the air is the difference between the force which compresses the air in the vessel and that which compresses the internal air, the expelling force is q; whence, the forces being as the quantities of motion,

$$p : p \times \frac{\text{D}' - \text{D}}{\text{D}} :: m\,v : n\,u,$$

m, n, being the masses expelled, v the velocity with which air rushes into a void, and u the velocity required. But the masses or number of particles which issue through the same orifice in

an instant, are as the densities and velocities conjointly; hence

$$m\,v : n\,u :: \text{D}\,v^2 : \text{D}'\,u^2,$$

and consequently,

$$p : p \times \frac{\text{D}' - \text{D}}{\text{D}} :: \text{D}\,v^2 : \text{D}'\,u^2,$$

which gives

$$u = v\sqrt{\frac{\text{D}' - \text{D}}{\text{D}'}}.$$

Moreover, from the proportion

$$p : p + q :: \text{D} : \text{D}',$$

we obtain

$$\text{D}' = \frac{\text{D}\,(p + q)}{p}$$

and

$$\text{D}' - \text{D} = \frac{\text{D}\,(p + q)}{p} - \text{D}$$

$$= \frac{\text{D}\,(p + q) - \text{D}\,p}{p} = \frac{\text{D}\,q}{p}.$$

Whence

$$\frac{\text{D}' - \text{D}}{\text{D}'} = \frac{\frac{\text{D}\,q}{p}}{\frac{\text{D}\,(p + q)}{p}} = \frac{q}{p + q}.$$

Substituting this value of $\frac{\text{D}' - \text{D}}{\text{D}'}$ in the above expression for u, we have the following simple and convenient formula, namely,

$$u = v\sqrt{\frac{q}{p + q}}.$$

494. We have taken no notice of the effect of the air's elasticity upon the velocity of influx into a void. Let $ABCD$ be a vessel containing air of any density D. This air is in a state of compression, and if the compressing force be removed it will expand, and its elasticity will diminish with the density. Now its elasticity, in whatever state, is measured by the force which

keeps it in that state; and the force which keeps common air at its ordinary density is the pressure of the atmosphere, and equivalent to that of a column of mercury 30 inches in height. If, therefore, we suppose this air, instead of being confined by the top of the vessel, to be pressed down by a movable piston carrying a cylinder of mercury of the same base and 30 inches high, its elasticity will balance this pressure just as it does the pressure of the atmosphere; and since from its fluidity the pressure received on any one part is propagated through every part and in every direction, it will press on any small portion of the vessel by its elasticity, as when loaded with this column. Hence, if this small portion of the vessel be removed leaving an opening into the void, the air will begin to flow out with the same velocity as it would flow out when pressed by its own weight only, or with the velocity acquired by falling from the top of a homogeneous atmosphere, or 1338 feet per second. But as soon as a portion of air had passed through the orifice, the density of that remaining in the vessel being reduced, its elasticity, and consequently the expelling force, is diminished. But the matter to be moved is diminished in the same proportion as the density, the capacity of the vessel remaining unchanged; therefore, since the density and elasticity follow the same law, the mass moved will vary as the moving force, and the velocity will continue the same from the beginning to the end of the efflux.

495. The velocity with which the air issues out of a vessel under the circumstances above supposed, being constant, we can readily compare the velocity given by the theory with that
Fig. 239. found by experiment. Let A be a cask of known capacity in the top of which is an aperture a of a known area. The tube TB, recurved at B, is soldered or screwed into the top of the cask. The aperture a is stopped while water is poured into the tube T till it is full, at which time a quantity of water will have passed out at B condensing the air in the cask till its spring is equal to the weight of the water in the tube. At this time a cock placed over the tube T, sufficiently large to supply water as fast as it can descend into the vessel A, is to be opened to keep the tube constantly filled. For this purpose one person must constantly tend it, while another opens the aperture a, which needs only to be closed with the finger, the seconds being

counted from the moment the finger is removed till the water flies out at a. Hence, knowing the capacity of the vessel and the area of the aperture, we obtain the velocity. If the tube TB should be continued nearly to the bottom of A, while A was filling with water, the length of the compressing column would be gradually diminished, and consequently the pressure constantly changing. To avoid any irregularity from this cause the open end of the tube is placed as near the top of the cask as is consistent with a free passage for the water.

The vessel was made to contain $15^{\text{lb.}}\ 6^{\text{oz.}}$ of water, from which its capacity is found to be 425,088 cubic inches. The area of the aperture a through which the water is discharged was 0,0046 inches.

(1.) The altitude of T above the cask being 30 inches, the time of expelling the air was found by several trials to be 33″.

(2.) The altitude of T being 6 feet, the time of expelling the air was 21,3″.

In the first experiment, 425,088, the capacity of the cask, being divided by 0,0046, the area of the aperture, gives 92410,4 inches for the length of the stream continued during 33″. Hence $\frac{92410{,}4}{12 \times 33} = 233{,}3$ feet, the velocity per second.

From the second experiment we deduce by a similar process, 361,6 for the velocity per second; and to show the correspondence of this with the first, we use the proportion

$$\sqrt{2\tfrac{1}{2}} : \sqrt{6} :: 233{,}3 : 361{,}8,$$

differing from the experimental result one fifth of a foot.

496. To compare the velocity thus found by experiment with that assigned by theory, we use the proportion

$$\sqrt{6} : \sqrt{34} :: 361{,}6 : 860{,}5,$$

the velocity with which the atmosphere would begin to enter a void. Taking the result before found, namely, 1338, and multiplying it by 0,63, agreeably to what is laid down in article 479,

we shall have 842,94, differing from the experimental result about $\frac{1}{50}$ part.

497. Let it be proposed to find the quantity of air expelled
Fig. 238. into an infinite void from the aperture C of the vessel $ABCD$ during any time t, and the density of the remaining air at the end of that time.

The bulk expelled during the instant $d\,t$ will be $\sigma\,d\,t\,\sqrt{2\,g\,h}$, the velocity $\sqrt{2\,g\,h}$ being constant, and consequently the quantity will be $\sigma\,\text{D}'\,d\,t\,\sqrt{2\,g\,h}$. The quantity at the beginning of the efflux is $b\,\text{D}$, b being as before the bulk of the vessel; and when the air has acquired the density D', the quantity in the vessel is $b\,\text{D}'$, and the quantity expelled is $b\,\text{D} - b\,\text{D}'$; consequently, the quantity discharged during the instant $d\,t$ must be the differential of $b\,\text{D} - b\,\text{D}'$, that is, $-\,b\,d\,\text{D}'$. Hence we have the equation

$$\sigma\,\text{D}'\,d\,t\,\sqrt{2\,g\,h} = -\,b\,d\,\text{D}',$$

and

$$d\,t = -\frac{b\,d\,\text{D}'}{\sigma\,\text{D}'\,\sqrt{2\,g\,h}} = \frac{b}{\sigma\,\sqrt{2\,g\,h}} \times -\frac{d\,\text{D}'}{\text{D}'},$$

the integral of which is

$$t = \frac{-\,b}{\sigma\,\sqrt{2\,g\,h}} \times \text{h. log. D}' + \text{C},$$

h. log. denoting the hyperbolic or Naperian logarithm of D'. When $t = 0$, D' is equal to D; whence

$$\text{C} = \frac{b'}{\sigma\,\sqrt{2\,g\,h}}\,\text{h. log. D};$$

therefore,

$$t = \frac{b}{\sigma\,\sqrt{2\,g\,h}}\,\text{h. log.}\,\frac{\text{D}}{\text{D}'} = \frac{b}{8\,\sigma\,\sqrt{h}}\,\text{h. log.}\,\frac{\text{D}}{\text{D}'},$$

nearly.

It is obvious that no finite time will be sufficient for the vessel to empty itself; for, as D' must in this case be equal to zero, $\frac{\text{D}}{\text{D}'}$ will be infinite, and its logarithm will also be infinite; so that t will be infinite.

498. It is by a train of reasoning precisely similar, that we ascertain the quantity of condensed air which will make its escape from a vessel into the atmosphere in a given time. Let Δ be the density of the condensed air, and h' the height of a homogeneous atmosphere corresponding to it; also let D be the density of the atmosphere. The air, having for its density Δ, will obviously, when mixing with the atmosphere, have the same velocity as though it were rushing into a vacuum with the density $\Delta - \text{D}$. Now the height of a homogeneous fluid corresponding to this density is found by the proportion

$$\Delta : \Delta - \text{D} :: h' : h' \frac{\Delta - \text{D}}{\Delta}.$$

From the equation $v = \sqrt{2 g s}$, it will be seen that the velocities acquired by falling from different heights are proportional to the square roots of these heights. If, therefore, v be taken to represent the velocity of common air rushing into a vacuum, and h the height of the corresponding homogeneous atmosphere, we shall have, u being the velocity belonging to the height $h' \frac{\Delta - \text{D}}{\Delta}$,

$$h : h' \frac{\Delta - \text{D}}{\Delta} :: v^2 : u^2 = v^2 \frac{h'}{h} \cdot \frac{\Delta - \text{D}}{\Delta};$$

whence

$$u = v \sqrt{\frac{h'}{h} \cdot \frac{\Delta - \text{D}}{\Delta}}.$$

Moreover the proportion

$$\text{D} : \Delta :: h : h',$$

gives

$$\frac{h'}{h} = \frac{\Delta}{\text{D}};$$

therefore

$$u = v \sqrt{\frac{\Delta}{\text{D}} \cdot \frac{\Delta - \text{D}}{\Delta}} = v \sqrt{\frac{\Delta - \text{D}}{\text{D}}}.$$

Let Δ' be the density of the condensed air after the time t, b being, as before, the bulk of the vessel, and σ a section of the

aperture, we shall have

$$\sigma\, v\, \Delta'\, d\, t \sqrt{\frac{\Delta' - \text{D}}{\text{D}}} = - b\, d\, \Delta',$$

and

$$d\, t = \frac{- b\, d\, \Delta'}{\sigma\, v\, \Delta' \frac{\sqrt{\Delta' - \text{D}}}{\sqrt{\text{D}}}} = \frac{- b \sqrt{\text{D}}}{\sigma\, v} \times \frac{d\, \Delta'}{\Delta' \sqrt{\Delta' - \text{D}}}.$$

Let $\sqrt{\Delta' - \text{D}} = x$, then $\Delta' - \text{D} = x^2$, and $\Delta' = x^2 + \text{D}$, from which we have

$$d\, \Delta' = 2\, x\, d\, x,$$

and hence

$$\frac{d\, \Delta'}{\Delta' \sqrt{\Delta' - \text{D}}} = \frac{2\, x\, d\, x}{(x^2 + \text{D})\, x} = \frac{2\, d\, x}{x^2 + \sqrt{\text{D}^2}}.$$

Cal. 120. Now the integral of $\frac{2\, d\, x}{x^2 + \sqrt{\text{D}^2}}$ is equal to $\frac{1}{\sqrt{\text{D}}}$ multiplied by an arc whose tangent is $\frac{x}{\sqrt{\text{D}}}$, radius being 1.

Whence

$$t = \frac{- 2\, b \sqrt{\text{D}}}{\sigma\, v} \times \frac{1}{\sqrt{\text{D}}} \times \text{an arc whose tang. is } \frac{\sqrt{\Delta' - \text{D}}}{\sqrt{\text{D}}} + \text{C}.$$

When

$$t = 0,\ \Delta' \text{ is equal to } \Delta,$$

and

$$\text{C} = \frac{2\, b}{\sigma\, v} \times \text{an arc whose tang. is} \sqrt{\frac{\Delta - \text{D}}{\text{D}}}.$$

Let the former arc be represented by a and the latter by a'; we shall have

$$t = \frac{2\, b}{\sigma\, v}\, (a' - a).$$

When the time is required in which the density of the air contained in the vessel shall be reduced to that of the external atmosphere; as $\Delta' = \text{D}$, in this case, it follows that $a = 0$, and

$$t = \frac{2\, b\, a'}{\sigma\, v}.$$

To illustrate this by an example, let it be required to find the time in which air of double the atmospheric density, confined in a cubical vessel, each of whose sides is 12 feet, will expand into the atmosphere through an opening of one tenth of an inch in diameter, so as to be reduced to the common density. The above formula gives, by substitution,

$$t = \frac{2 \times 12^3}{0{,}007854 \times 1338} \times \text{an arc whose tang. is } 1,$$

$$= \frac{2 \times 12^3 \times 100}{0{,}7854 \times 1338} \times \text{an arc whose tang. is } 1,$$

$$= \frac{2 \times 12^3 \times 100}{1338} = 258'' = 4'\ 18''.$$

499. Even although the density of the confined air were infinitely greater than that of the atmosphere, the time in which it would be reduced would be a finite quantity. For Δ being infinitely greater than D, the

$$\text{tang.}\ \sqrt{\frac{\Delta - \text{D}}{\text{D}}}$$

is also infinite, and corresponds to an arc of 90°. Hence in this case we have

$$t = \frac{2\,b}{\sigma\,v} \times 2 \times 0{,}7854.$$

The capacity of the vessel and the area of the aperture being the same as in the last example, we should have for the time in which this infinitely condensed air would be reduced to the same density with the atmosphere

$$t = \frac{2 \times 1728}{0{,}007854 \times 1338} \times 2 \times 0{,}7854 = 561'' = 8'\ 36''.$$

Of the Resistance of Fluids to Bodies moving in them.

500. The force with which solid bodies moving in fluids, as water, air, &c., are impeded and retarded, is usually termed the *resistance* of fluids; and as all our machines move either in water or in air, or both, it becomes a matter of importance in the theory of mechanics to inquire into the nature of this kind of force.

We know by experience that force must be applied to a body in order that it may move through a fluid, such as air or water; and that a body projected with any velocity is gradually retarded in its motion, and generally brought to rest. We also know that a fluid in motion will hurry a solid body along with it, and that force is necessary to maintain the body in its place. And as our knowledge of nature teaches us that the mutual actions of bodies are in every case equal and opposite, and that the observed change of motion is the only indication and measure of the changing force, we infer that the force which is necessary to keep a body immovable in a stream of water, flowing with a certain velocity, is the same with that which is required to move this body with an equal velocity through stagnant water.

A body in motion appears to be resisted by a stagnant fluid, because it is a law of mechanical nature that force must be employed in order to put any body in motion. Now, the body cannot move forward without putting the contiguous fluid in motion, and force is to be used to produce this motion. In like manner, a quiescent body is impelled by a stream of fluid, because the motion of the contiguous fluid is diminished by this solid obstacle; the resistance, therefore, or impulse, differs in no respect from the ordinary communication of motion among solid bodies, at least in its nature; although it may be far more difficult to reduce the various circumstances attending it to accurate computation, or to obtain all the requisite data on which to found the calculation.

501. The resistance which a body suffers from the fluid medium through which it is impelled depends on the velocity, form, and magnitude of the body, and on the inertia and tenacity of the fluid. For fluids resist the motion of bodies through them, (1.) by the inertia of their particles; (2.) by their tenacity, that is, the adhesion of those particles; (3.) by the friction of the body against the particles of the fluid. In perfect fluids the two last causes of resistance are very inconsiderable, and therefore are not taken into the account; but the first is always very considerable, and obtains equally in the most perfect as in the most imperfect fluids. And that the resistance varies with the velocity, shape, and magnitude of the moving body is sufficiently obvious.

We must carefully distinguish between *resistance* and *retardation;* resistance is the quantity of *motion*, retardation the quantity of *velocity*, which is lost; therefore, the retardations are as the resistances applied to the quantities of matter; and in the same body the resistance and retardation are proportional.

502. *To determine the force of fluids in motion, or the resistance of fluids against bodies moving in them.*

(1.) In fluids uniformly tenacious, the resistance is as the velocity with which the body moves. For, since the cohesion of the particles of the fluid is always the same for the same space, whatever be the velocity, the resistance from this cohesion will be as the space described in a given time; that is, as the velocity.

(2.) In a fluid whose particles move freely without disturbing each others' motions, and which flows in behind as fast as a plane body moves forward, so that the pressure on every part of the body is the same as if the body were at rest, the resistance will be as the density of the fluid.

(3.) On the same hypothesis the resistance will be as the square of the velocity. For the resistance must vary as the number of particles which strike the plane in a given time, multiplied into the force of each against the plane; but both the number and the force is as the velocity, and consequently the resistance is as the square of the velocity.

This proof supposes that after the body strikes a particle, the action of that particle entirely ceases; whereas the particles, after they are struck, must necessarily diverge, and act upon the particles behind them; thus causing some difference between theory and experiment. This hypothesis, however, on account of its simplicity, is generally retained, and corrected afterwards by deductions from actual experiments.

This ratio of the square of the velocity may be otherwise derived, thus.

It is evident, that the resistance to a plane, moving perpendicularly through an infinite fluid, at rest, is equal to the pressure or force of the fluid on the plane at rest, the fluid moving with the same velocity, and in the contrary direction to that of the plane in the former case. But the force of the fluid in motion must be equal to the weight or pressure which generates that motion; and which, it is known, is equal to the weight or pressure of a column of the fluid, whose base is equal to the plane, and its altitude equal to the height through which a body must fall by the force of gravity, to acquire the velocity of the fluid; and that altitude is, for the sake of brevity, called the altitude due to the velocity. So that if σ denote the surface of the plane, v the velocity, and s the specific gravity of the fluid; then the altitude due to the velocity v being $\frac{v^2}{2\,g}$, the whole resistance or moving force m, will be

$$\sigma \times \mathrm{s} \times \frac{v^2}{2g} = \frac{\sigma\, \mathrm{s}\, v^2}{2\,g};$$

g being 32,2 feet. And hence, other things being the same, the resistance is as the square of the velocity.

(4.) If the direction of the motion, instead of being perpendicular to the plane, as above supposed, be inclined to it at any angle, then the resistance to the plane in the direction of the motion, as assigned above, will be diminished in the triplicate ratio of radius to the sine of the angle of inclination, or in the ratio of 1 to i^3, where i is the sine of the inclination.

For AB being the direction of the plane, and BD that of the motion, ABD the angle whose sine is i; the number of particles or quantity of the fluid which strikes the plane will be diminished in the ratio of 1 to i; and the force of each particle will likewise be diminished in the same ratio; so that on both these accounts the resistance will be diminished in the ratio of 1 to i^2; that is, in the duplicate ratio of radius to the sine of ABD. But further, it must be considered that this whole resistance is exerted in the direction BE perpendicular to the plane; and any force in a direction BE is to its effect in a direction AE, parallel to BD, as AE to BE, or as 1 to i. Consequently, on all these accounts, the resistance in the direction of the motion is diminished in the ratio of 1 to i^3. And if this be compared with the result of the preceding step, we shall have for the whole resistance, or the moving force, on the plane, Fig. 240.

$$m = \frac{\sigma s v^2 i^3}{2g}.$$

(5.) If w represent the weight of the body whose plane surface σ is resisted by the absolute force m, then the retarding force

$$f = \frac{m}{w} = \frac{\sigma s v^2 i^3}{2gw}.$$

(6.) And if the body be a cylinder whose surface or end is σ, and diameter D, or radius R, moving in the direction of its axis; then, because $i = 1$, and $\sigma = \pi R^2 = \frac{1}{4}\pi D^2$, where $\pi = 3{,}141593$, the resisting force m will be

$$= \frac{\pi s D^2 v^2}{8g} = \frac{\pi s R^2 v^2}{2g};$$

and the retarding force

$$f = \frac{\pi s D^2 v^2}{8gw} = \frac{\pi s R^2 v^2}{2gw}.$$

(7.) This is the value of the resistance when the end of the cylinder is a plane perpendicular to its axis, or to the direction of the motion. But were its face a conical surface, or an elliptic section, or any other figure every where equally inclined to the axis, the sine of the inclination being i; then the number of particles of the fluid striking the surface being still the same, but the force of each, opposed to the direction of the motion, diminished

in the duplicate ratio of the radius to the sine of the inclination, the resisting force m would be

$$\frac{\pi \text{ S } D^2 v^2 i^2}{8 g} = \frac{\pi \text{ S } R^2 v^2 i^2}{2 g}.$$

But if the body were terminated by an end or surface of any other form, as a spherical one, where every part of it has a different inclination to the axis; then a further investigation becomes necessary.

503. *To determine the resistance of a fluid to any body moving in it, having a curved end as a sphere, a cylinder with a hemispherical end, &c.*

Fig. 241. (1.) Let BEAD be a section through the axis CA of the solid, moving in the direction of that axis. To any point of the curve draw the tangent EG, meeting the axis produced in G; also draw the perpendicular ordinates EF, $e f$, indefinitely near to each other; and draw $a e$ parallel to CG.

Putting CF $= x$, EF $= y$, BE $= z$, $i =$ sine of the angle G, radius being 1; then $2 \pi y$ is the circumference whose radius is EF, or the circumference described by the point E, in revolving about the axis CA; and $2 \pi y \times$ E e, or $2 \pi y \, d z$, is the differential of the surface, or it is the surface described by E e, in its revolution about CA; hence

$$\frac{\text{S } v^2 i^3}{2 g} \times 2 \pi y \, d z, \text{ or } \frac{\pi \text{ S } v^2 i^3}{g} \times y \, d z$$

is the resistance on that ring, or the differential of the resistance to the body, whatever the figure of it may be; the integral of which will be the resistance required.

(2.) In the case of a spherical shape; putting the radius CA or CB $=$ R, we have

$$y = \sqrt{(R^2 - x^2)}, \quad i = \frac{EF}{EG} = \frac{CF}{CE} = \frac{x}{R},$$

and $$y \, d z \text{ or } EF \times E \, e = CE \times a \, e = R \, d x;$$

therefore the general differential

$$\frac{\pi \text{ S } v^2}{g} \cdot i^3 y \, d z$$

becomes

$$\frac{\pi \text{ s } v^2}{g} \cdot \frac{x^3}{\text{R}^3} \cdot \text{R}\, d\, x = \frac{\pi \text{ s } v^2}{g\, \text{R}^2} \cdot x^3\, d\, x;$$

the integral of which, or $\frac{\pi \text{ s } v^2}{4\, g\, \text{R}^2}\, x^4$, is the resistance to the spherical surface generated by BE. And when x or CF is $=$ R or CA, it becomes $\frac{\pi \text{ s } v^2\, \text{R}^2}{4\, g}$ for the resistance on the whole hemisphere; which is also equal to $\frac{\pi \text{ s } v^2\, \text{D}}{16\, g}$, where $\text{D} = 2\, \text{R}$, the diameter.

(3.) But the perpendicular resistance to the circle of the same diameter D or BD, by section 6 of the preceding problem, is $\frac{\pi \text{ s } v^2\, \text{D}^2}{8\, g}$; which being double the former, shows that *the resistance to the sphere is just equal to half the direct resistance to a great circle of it, or to a cylinder of the same diameter.*

(4.) Since $\frac{1}{6}\, \pi\, \text{D}^3$ is the magnitude of the globe; if s′ denote its density or specific gravity, its weight w will be $= \frac{1}{6}\, \pi\, \text{D}^3\, \text{s}'$, and therefore the retarding force

$$f \text{ or } \frac{m}{w} = \frac{\pi \text{ s } v^2\, \text{D}^2}{16\, g} \cdot \frac{6}{\pi\, \text{s}'\, \text{D}^3} = \frac{3 \text{ s } v^2}{8\, g\, \text{s}'\, \text{D}};$$

which is also equal to

$$\frac{v^2}{2\, g\, s}.$$

Hence

$$\frac{3 \text{ s}}{4\, \text{s}'\, \text{D}} = \frac{1}{s},$$

and

$$s = \frac{\text{s}'}{\text{s}} \cdot \tfrac{4}{3}\, \text{D};$$

which is the space that would be described by the globe while its whole motion is generated or destroyed by a constant force equal to the resistance, if no other force acted on the globe to continue its motion. And if the density of the fluid were equal to that of the globe, the resisting force sufficient would be acting constantly on the globe without any other force, to generate or destroy the motion while describing the space $\frac{4}{3}$ D, or $\frac{4}{3}$ of its diameter.

(5.) Hence the greatest velocity that a globe acquires in descending through a fluid, by means of its relative weight in the fluid, will be found by putting the resisting force equal to that weight. For, after the velocity has arrived at such a degree that the resisting force is equal to the weight that urges it, it will increase no longer, and the globe will afterwards continue to descend with that velocity uniformly. Now, s′ and s being the specific gravities respectively of the globe and fluid, s′ — s will be the relative gravity of the globe in the fluid, and therefore

$$w = \tfrac{1}{6} \pi \text{D}^3 (\text{s}' - \text{s})$$

is the weight by which it is urged; also

$$m = \frac{\pi \text{ s } v^2 \text{D}^2}{16\, g}$$

is the resistance; consequently

$$\frac{\pi \text{ s } v^2 \text{D}^2}{16\, g} = \tfrac{1}{6} \pi \text{D}^3 (\text{s}' - \text{s}),$$

when the velocity becomes uniform; from which equation is found

$$v = \sqrt{\left(2\, g \cdot \tfrac{4}{3} \text{D} \cdot \frac{\text{s}' - \text{s}}{\text{s}}\right)},$$

for the above uniform motion or greatest velocity.

By comparing this with the general equations, $v = \sqrt{g\, s}$, it will be seen that the greatest velocity is that acquired by the accelerating force $\frac{\text{s}' - \text{s}}{\text{s}}$, in describing the space $\frac{4}{3}$ D, or that acquired by gravity in describing freely the space

$$\frac{\text{s}' - \text{s}}{\text{s}} \cdot \tfrac{4}{3} \text{D}.$$

If s′ = 2 s, or the specific gravity of the globe be double that of the fluid, $\frac{\text{s}' - \text{s}}{\text{s}} = 1$ = the natural force of gravity; and the globe will attain its greatest velocity in describing $\frac{4}{3}$ D or $\frac{4}{3}$ of its diameter. It is further evident, that if the globe be very small, it will soon attain its greatest velocity, whatever its density may be.

If a leaden ball, for example, one inch in diameter descend in water, and in air of the usual density at the earth's surface, the specific gravities of these bodies being 11,3, 1, 0,00122, respectively, since $\frac{4}{3}$ of an inch is $\frac{4}{36}$, or 0,11, of a foot, $\frac{s' - s}{s}$ becomes 10,3 in the case of water, and

$$\frac{11,3 - 0,00122}{0,00122} = 926 \text{ nearly,}$$

in the case of air, we shall have

$$v = \sqrt{2 \times 32,2 \times 0,11 \times 10,3} = 8,54$$

nearly, for the greatest velocity the ball can acquire per second in water; and

$$\sqrt{2 \times 32,2 \times 0,11 \times 926} = 256$$

nearly, for the greatest velocity in air.

But if the globe were only one hundredth of an inch in diameter, the greatest velocities would be only $\frac{1}{10}$ of the above results, or 0,85 of a foot in water, and 25,6 in air; and if the ball were still further diminished, the greatest velocity would be diminished also, in the subduplicate ratio of the diameter of the ball. This is well illustrated in the fall of rain. The different sized drops descend with different degrees of rapidity, but all so gently as to cause no injury. Were this fluid so constituted as to allow the drops to form in larger masses, or were the air much less dense, tender vegetables would suffer by the shock, as they sometimes do in fact by the more rapid descent of hail.

504. It appears from the third step of the preceding article, that the resistance to the motion of a cylinder moving in the direction of its axis is double that of a globe of the same diameter; and in experiments with bodies that move slowly, this will nearly hold true in water, but still more nearly in air; because its particles move more freely than those of water, and less disturb each others' motions; but when the motion is more rapid, considerable aberrations will occur, both from the mutual disturbance of the particles, and from the fluid not flowing in so fast behind as the body moves forward; in the air also, a new cause of deviation will arise, from the condensation of the fluid before

the body. Sir Isaac Newton supposes, that in a continuous non-elastic fluid, infinitely compressed, the resistances of a sphere and cylinder of equal diameters are equal; but this appears to be an error in theory as well as in fact. When the motion is slow in water, the fluid may be conceived to be nearly of the nature which Newton supposes; yet the resistances are almost as coincident with theory as when the motion is in air; thus M. Borda found the resistance of a sphere moving in water to be to that of its greatest circle as 1 to 2,508, and in air the resistances were as 1 to 2,45. The experiments of Dr. Hutton in air give the resistances at a mean as 1 to $2\frac{1}{3}$.

The reason that experiment gives the ratio of the resistances greater than that of 2 to 1 seems to be this; in theory it is supposed that the action of every particle of the fluid ceases the instant it makes its impact on the solid; but this is not actually the case, as we have before observed; and since the particles, after impact on the sphere, slide along the curved surface, and hence escape with more facility than along the face of the cylinder, the error will be greater in the cylinder; that is, the greater resistance will exceed the theory more than the less. It is also to be observed, that the difference between the resistances of the globe and cylinder in water is greater than in air; and this is directly contrary to what might be inferred from Newton's reasoning, which supposes them equal in a continuous fluid, but in the ratio of 1 to 2 in a rare fluid.

505. *To determine the relations of velocity, space, and time, of a ball moving in a fluid in which it is projected with a given velocity.*

Let $u =$ the velocity of projection, s the space described in any time t, and v the velocity acquired. Now, by step 4, article 503, the accelerating force $f = \frac{3\,\text{s}\,v^2}{8\,g\,\text{s}'\,\text{D}}$; where s$'$ is the density of the ball, s that of the fluid, and D the diameter of the
282. ball. Therefore the general equation $v\,d\,v = g\,f\,d\,s$ becomes

$$v\,d\,v = \frac{-3\,\text{s}\,v^2}{8\,\text{s}'\,\text{D}}\,d\,s;$$

and hence

$$\frac{d\,v}{v} = \frac{-3\,\text{s}}{8\,\text{s}'\,\text{D}}\,d\,s = -c\,d\,s,$$

putting c for $\frac{3\,\text{s}}{8\,\text{s}'\,\text{D}}$. The correct integral of this is log. u — log. v, or log. $\frac{u}{v} = c\,s$. Or putting $e = 2{,}718281828$, the number whose hyp. log. is 1, then $\frac{u}{v} = e^{cs}$, and the velocity

$$v = \frac{u}{e^{cs}} = ue^{-cs}.$$

506. The velocity v at any time being the e^{-cs} part of the first velocity, the velocity lost in any time will be the $1 - e^{-cs}$ part, or the $\frac{e^{cs} - 1}{e^{cs}}$ part of the first velocity.

(1.) If a globe, for example, be projected with any velocity in a medium of the same density with itself, and it describe a space equal to 3 D or 3 of its diameters; then $s = 3\,\text{D}$, and

$$c = \frac{3\,\text{s}}{8\,\text{s}'\,\text{D}} = \frac{3}{8\,\text{D}};$$

therefore $c\,s = \frac{9}{8}$, and the velocity lost is

$$\frac{e^{cs} - 1}{e^{cs}} = \frac{2{,}08}{3{,}08},$$

or nearly $\frac{2}{3}$ of the projectile velocity.

(2.) If an iron ball of two inches diameter were projected with a velocity of 1200 feet per second; to find the velocity lost after moving through any space, as 500 feet of air; we should have

$$\text{D} = \tfrac{2}{12} = \tfrac{1}{6},\ u = 1200,\ s = 500,\ \text{s}' = 7\tfrac{1}{3},\ \text{s} = 0{,}0012;$$

and therefore

$$c\,s = \frac{3\,\text{s}\,s}{8\,\text{s}'\,\text{D}} = \frac{3 \,.\, 0{,}0012 \,.\, 500}{8 \,.\, 7\frac{1}{3} \,.\, \frac{1}{6}} = \frac{3 \,.\, 12 \,.\, 500 \,.\, 3 \,.\, 6}{8 \,.\, 22 \,.\, 10000} = \frac{81}{440},$$

and

$$v = \frac{1200}{e^{\frac{81}{440}}} = 998 \text{ feet per second};$$

having lost 202 feet, or nearly $\frac{1}{6}$ of its first velocity.

(3.) If the earth revolved about the sun, in a medium as dense as the atmosphere near the earth's surface; and it were required to find the quantity of motion lost in a year; since the earth's mean density is about $4\frac{1}{2}$, and its distance from the sun 12000 of its diameters, we have

$$24000 \times 3{,}1416 = 75398 \text{ diameters} = s,$$

and

$$c\,s = \frac{3 \,.\, 75398 \,.\, 12 \,.\, 2}{8 \,.\, 10000 \,.\, 9} = 7{,}5398\,;$$

hence $\frac{e^{cs} - 1}{e^{cs}} = \frac{1880}{1881}$ parts are lost of the first motion in the space of a year, and only the $\frac{1}{1881}$ part remains.

(4.) To find the time t; we have

230. $$d\,t = \frac{d\,s}{v} = \frac{d\,s}{\frac{u}{e^{cs}}} = \frac{e^{cs}\;d\,s}{u}.$$

Now, to find the integral of this, put $z = e^{cs}$; then is $c\,s = \log.\ z$, and

$$c\,d\,s = \frac{d\,z}{z}, \text{ or } \; d\,s = \frac{d\,z}{c\,z};$$

consequently

$$d\,t \text{ or } \frac{e^{cs}\;d\,s}{u} = \frac{z\,d\,z}{u} = \frac{d\,z}{u\;c};$$

and hence

$$t = \frac{z}{u\;c} = \frac{e^{cs}}{u\;c} + \text{C}.$$

But as t and s vanish together, and when $s = 0$, the quantity

$$\frac{e^{cs}}{u\;c} \text{ is equal to } \frac{1}{u\;c};$$

therefore

$$t = \frac{e^{cs} - 1}{u\;c} = \frac{1}{c\,v} - \frac{1}{c\,u} = \frac{1}{c}\left(\frac{1}{v} - \frac{1}{u}\right)$$

the time sought; where $c = \frac{3\,s}{8\,s'\,D}$, and $v = \frac{u}{e^{cs}}$ the velocity.

On the Theory of the Air-Pump, and Pumps for raising Water.

507. The *Air-Pump* is a machine fitted to exhaust the air from a proper vessel, and thus to produce what is called a vacuum; it is one of the most useful of philosophical experiments. By means of it the chief propositions relative to the weight and elasticity of the air are proved experimentally, in a simple and satisfactory manner.

EFGH represents a square table of wood, *A*, *A* two strong barrels or tubes of brass, firmly retained in their position by the cross-piece *TT*, which is pressed on them by screws *O*, *O*, fixed on the tops of the brass pillars *N*, *N*. These barrels communicate with a cavity in the lower part *D* of the table. At the bottom within each barrel is fixed a valve, opening upwards; and in each barrel a piston works, having a valve likewise opening upwards. The pistons are moved by a cog-wheel in the piece *TT*, turned by the handle *B*, of which wheel the teeth catch in the racks of the pistons *C*, *C*. *PQ* is a circular brass plate, having near its centre the orifice *K* of a concealed pipe that communicates with the cavity at *D*; at *V* is a screw that closes the orifice of another pipe, and which is turned for the purpose of admitting the external air when required. *LM* is a glass vessel, from which the air is to be exhausted, and which has obtained the name of *receiver*, because it receives or holds the subjects on which the experiments are to be made. This receiver is placed on the plate *PQ*, and is accurately fitted to it by grinding, or by means of moistened or oiled leather. Fig. 244.

When the handle *B* is turned, one of the pistons is raised and the other depressed; consequently a void space is left between the raised piston and the lower valve in the corresponding barrel; the air contained in the receiver *LM* communicating with the barrel by the orifice *K* immediately raises the lower valve by its spring, and expands into the void space; and thus a part of the air in the receiver is extracted. The handle then, being turned the contrary way, raises the other piston, and performs the same operation in the barrel containing it; while in the mean time the

first mentioned piston being depressed, the air by its spring closes the lower valve, and raising the valve in the piston makes its escape. The motion of the handle being again reversed, the first barrel again exhausts, while the second discharges the air in its turn; and thus during the whole time the pump is worked, one barrel exhausts the air from the receiver, while the other discharges it through the valve in its piston. Hence it is evident, that the air can never be entirely exhausted; for it is the elasticity of the air in the receiver that raises the valve, and forces it into the barrel *; and each operation can only take away a certain part of the remaining air, which is in proportion to the quantity before the stroke, as the capacity of the barrel to the sum of the capacities of the barrel, receiver, and communicating pipe.

508. Now if we suppose no vapor from moisture, &c., to rise in the receiver, the degree of exhaustion after any number of strokes of the piston may be determined by knowing the respective capacities of the barrel and of the receiver, including the pipe of communication, &c. For, as we have seen above, that every stroke diminishes the density in a constant proportion, namely, as much as the whole capacity exceeds that of the cylinder or barrel; the exhaustion will go on in a geometrical progression, the ratio of which is the same as that which the sum of the capacities of the receiver and barrel bears to that of the receiver; and this ratio of exhaustion will continue until the elasticity of the included air is so far diminished by its rarefaction as to render it too feeble to push up the valve of the piston.

Let then the capacity of the barrel, receiver, and pipe of communication together be expressed by $b + r$, and that of the

* Various contrivances have been adopted to facilitate the motion of the valve, and thus to allow the air, when in a state of great rarefaction, to pass from the receiver into the barrels. Smeaton had recourse to a valve presenting a broad surface, and supported on thin bars. Others have proposed to raise the valve mechanically by connecting it with the piston in such a manner that the piston shall exert its action at the moment it reaches the top of the barrel. A method suggested itself to Dr. Prince much more perfect and

barrel alone by b, and let 1 represent the primitive density of the air in the pump; we shall have

$$b + r : r :: 1 : \frac{r}{b + r} = \text{the density after 1 stroke of the piston.}$$

$$b + r : r :: \frac{r}{b + r} : \frac{r^2}{(b + r)^2} = \text{density after 2 strokes,}$$

$$b + r : r :: \frac{r^2}{(b + r)^2} : \frac{r^3}{(b + r)^3} = \text{density after three strokes;}$$

and the nth power of the ratio $\frac{r}{b + r}$,

or $$\frac{r^n}{(b + r)^n} = \text{D, the density after } n \text{ strokes.}$$

From which we may easily find the density after any number of strokes of the piston necessary to rarefy the air a number of times, or to give it a certain density D, the primitive density being 1. For the above equation, expressed logarithmically, is

$$n \times \log. \frac{r}{b + r} = \log. \text{D};$$

or

$$n \times (\log. r - \log. (b + r)) = \log. \text{D};$$

---

more simple. It dispenses with the valve entirely by extending the barrel downward so as to admit of the piston's descending below the opening which communicates with the receiver, and thus allowing a free introduction of air into the barrel. The air in this case is expelled through a valve at the top of the barrel, opening upward. There will still be a limit however to the exhaustion; for the air cannot be forced through the valve at the top of the barrel, unless its elasticity, and consequently its density, produced by the motion of the piston, exceeds the density of the external air. To remove this difficulty Dr. Prince enclosed the valve in question in a vessel furnished with a small exhausting apparatus, by which the valve was relieved from the pressure of the atmosphere. This appendage is not necessary, except for purposes of very accurate exhaustion.

consequently

$$n = \frac{\log.\ \text{D}}{\log.\ r - \log.\ (b + r)};$$

in which expression D will be a fraction. If the number of times which the air is rarefied be expressed by N, an integer, the logarithmic equation will be

$$n = \frac{\log.\ \text{N}}{\log.\ (b + r) - \log.\ r}.$$

A further reduction of the same theorem will furnish us with the proportion between the capacities of the receiver and barrel, when the air is rarefied to the density D′ by a definite number of strokes n of the piston. For since

$$\frac{r^n}{(b + r)^n} = \text{D}',$$

if we take the nth root of both members of the equation we shall have

$$\frac{r}{b + r} = \sqrt[n]{\text{D}'}.$$

Thus, if D′ be equal to $\frac{1}{177147}$, and the number of strokes $n = 11$, we shall find

$$\frac{\log.\ \text{D}'}{11} = \log.\ \tfrac{1}{3};$$

so that $r : b + r :: 1 : 3$, and $b : r :: 2 : 1$.

Pumps for Raising Water.

509. The term *pump* is generally applied to a machine for raising water by means of the pressure of the atmosphere. Of pumps there are a great many different sorts; we shall speak only of those in most common use, and shall give merely such a general description of their construction as will enable the student to understand the principles on which their operation depends.

The pumps most generally used are, the *sucking pump*, the *lifting pump*, and the *forcing pump;* these have some parts in common, and particularly the pistons and valves; they will therefore be treated in a connected way.

The *piston* is a body $ABCD$ of circular base, which may be moved through the interior part of the tube or body of the pump, filling it exactly as it moves along. The sucker or valve E is movable about a joint in such a manner as either to permit or to prevent the passage of the water, according as it presses upwards or downwards. In figures 245, 246, there are likewise valves in the pistons. $FGHK$ is another tube joined to the body of the pump, and is generally called the *pipe* or *sucking pipe;* its lower extremity is immersed in the water, of which we suppose RS to be the horizontal surface. Fig. 245.

510. The *sucking* pump is represented in figure 245. In this pump, if we suppose a power P applied to the handle of the piston so as to raise it from I to C, the air contained in the space $DVKHGFC$ tends by its spring to occupy the space that the piston leaves void; it therefore forces up the valve E, and enters into the body of the pump, its elasticity diminishing in proportion as it fills a greater space. Hence it will exert on the surface GH of the water a less effort than is made by the exterior air in its natural state upon the surrounding parts of the same surface RG, HS; and the excess of pressure on the part of the exterior air will cause the water to rise in the upper pipe GK to a certain height HN, such that the weight of the column GN, together with the spring of the superincumbent air shall just be a counterpoise to the pressure of the exterior air. At this time the valve E closes of itself; and if the piston be lowered, the air contained between the piston and the base IV of the body of the pump, having its density augmented as the piston is lowered, will at length have its density, and consequently its elasticity, greater than that of the exterior air; this difference of elasticity will constitute a force sufficient to push the valve L of the piston upwards, and some air will escape till the exterior and interior air are reduced to the same density. The valve L then falls again; and if we again elevate the piston, the water will be raised higher in $FGHK$, for the same reason as before. Thus,

after a certain number of strokes of the piston, the water will reach the body of the pump; where being once entered, it will be forced at each stroke of the piston through the spout X; for the water above the piston will then press upon the valve and keep it shut while the piston is rising; so that a cylinder of water whose height is equal to the stroke OT of the piston (or the vertical distance through which it passes) will be raised by each upward motion and forced through the aperture X, provided it is of an adequate magnitude.

511. The *lifting pump* is represented in figure 246. Its manner of operation is this; the piston PCD is here placed below the horizontal surface RS of the water, and when it is made to descend, it produces a vacuum between the valve E (which is pushed down by the exterior air) and the base CD of the piston. The weight of the water, together with that of the exterior air about R and S, presses up the valve L, and the water passes into the body of the pump; and when the water ceases to enter, the weight of the valve L closes it. Then, if the piston be raised, it raises all the water above it, forces up the valve E, and introduces the water into the part $IVYX$. When the piston is raised to its highest position, the valve E is made to close by the superincumbent water, and retains the fluid there until, by a fresh stroke of the piston, more water is forced upwards through the valve E; that which was before in the upper part of the pump being expelled through a proper orifice or spout in the neighbourhood of X, in order to make way for a new supply. By continuing the operation, water is delivered at every stroke of the piston.

512. The *forcing pump* unites in some measure the properties of the other two. The piston $ABCD$, which here has no valve, being elevated, rarefies the air in the space $DGHVOC$, and the water rises towards K; the subsequent descent of the piston forces some of the air in this space through the valve L; the next ascent of the piston closes the valve L, and raises the water in GK; and so on till the water passes through the valve E and enters the space $DIOC$. Then the piston being pushed down, closes the valve E, and some of the condensed air is forced through the valve L. A further stroke raises more water into

the space $DOIC$, and expels more air through L. At length the water reaches L, and the subsequent strokes raise it into the tube $MO\ m\ n$, whence it is carried off by a spout, as in the other pumps. Or, if this pump be closed at $m\ n$, excepting a narrow pipe PS, when the water is raised by the process just described to $o\ r$, above the bottom S of the tube, the elastic force of the compressed air in the space $n\ r\ o\ m$ will compel the water to issue from the aperture P in a continued stream or jet, thus forming a fire engine or artificial fountain.

513. Let us now enquire into the fundamental properties of these machines. By means of the *lifting* pump, water may be elevated to any height we please, provided we employ a sufficient force. But the estimate of this force requires various considerations. We must have regard to the dimensions of the piston, the barrel of the pump, the height to which the water is to be raised, and the velocity with which the water is to be moved, beside the effects of friction, &c. At present, however, we shall not examine these particulars in all their extent; but shall confine ourselves to one of them. Now it is certain that the power necessary to raise the water to any proposed height, must at least be capable of sustaining in equilibrium the pressure experienced by the base of the piston when it is kept at rest, and the fluid has attained the required height. This pressure, then, we proceed to estimate.

In general the power must be, at least, capable of sustaining the weight of a column of water which has for its base that of the piston, and for its altitude the distance between the surface RS of the water in the reservoir and the upper surface XY of that Fig. 246.
in the pump. For when the base DC of the piston is below the surface RS of the water in the reservoir, it is manifest that the power has not to sustain the pressure of the water contained between RS and DC; because this pressure is counterbalanced by that of the water surrounding the lower part of the pump, and which is transmitted by means of the inferior orifice of the pipe. The power, therefore, has only to sustain the pressure exerted upon the surface DC by the fluid comprehended between RS and XY; which pressure is equal to the weight of a column

of water whose base is CD and altitude the vertical distance be-
414. tween RS and XY.

When the piston is above $R' S'$, the surface of the water in the reservoir, then it is evident the water contained between DC and $R' S'$ does not press the piston downwards. But, as in this case it can only be sustained above $R' S'$ by the pressure of the air upon the water surrounding the pump, and as this pressure is only capable of sustaining in equilibrium the contrary pressure of the air upon the surface XY, it follows that the surface DC is charged with a weight equivalent to the column which has DC for its base and CR' for its altitude. And this pressure, joined to that which is exerted upon DC by the superincumbent fluid between DC and XY, makes the whole pressure upon the piston, as before, equal to that of a column of water whose base is DC, and height the distance between XY and $R' S'$.

514. The *sucking pump* requires in its theory the aid of other principles. We must enquire if under the proposed circumstances the water can possibly be raised to the piston, and made to pass through the valve L; for in some cases the water will never pass a certain altitude, how many strokes soever we give to the piston. To understand this, suppose that the water has been
Fig. 245. actually raised to T, and that the situation of the piston in the figure is the lowest which can be given to it; and, for greater simplicity, suppose that the pump is of the same internal diameter throughout. It is obvious that the air comprised in the space $CDTZ$ is of the same density and elasticity as the exterior air (at least leaving out of consideration the weight of the valve L, and the friction attending its motion;) for if its spring were less, the water would rise higher than ZT, and if it were greater, it would raise the valve L, and mix with the exterior air till both became of the same density. Suppose now that the play of the piston, or the distance through which it is raised at each stroke, is DO; then when the base CD is raised to OQ, the air which previously occupied the space $CDTZ$ will tend to expand and fill the space $QOTZ$; and, if the water did not rise, would actually be so expanded. Its elastic force would then be less than that of the natural air, in the ratio of $CDTZ$ to $QOTZ$, or of DT to OT.
468. If, therefore, this elastic force, together with the weight of the col-

umn of water whose height is ZR, constitute a pressure equal to that of the atmosphere, or equal to the weight of a column of water of the same base, and at a mean 34 feet in height, there will be an equilibrium, and the water will not rise further; if this 466. joint pressure is greater than that of 34 feet of water, the water cannot be retained so high; but if it is less than a column of 34 feet, the water will continue to rise in the pump.

515. From these considerations we may readily investigate a general theorem.

Let a be the altitude or vertical distance from the point O to the surface RS of the water in the reservoir, $p = OD$, the play of the piston, and x the distance OT; then we have $DT = x - p$, and ST, the height of the point T, will be $a - x$. Since the air contained in $CDTZ$ has the same density and elasticity as the exterior air, its force may be measured by a column of water of the same base ZT and 34 feet high; and because when this air is so expanded as to fill the space $QOTZ$, the elastic force will be less in the ratio of DT to OT, we shall have (rejecting the base of the column, as equally affecting every part of the process) this latter force expressed by the fourth term of this proportion,

$$x : x - p :: 34 : \frac{34}{x}(x - p).$$

But the force which the water, comprehended between ZT and RS, exerts in opposition to the exterior pressure of the air, is measured by the height $a - x$; consequently, the elastic force of the air in the space $QOTZ$, together with the weight of the water between ZT and RS, will be expressed by

$$\frac{34(x - p)}{x} + a - x.$$

Now in order that the water may always rise, this joint pressure must be less than the weight of a column of water 34 feet high by some variable quantity, which we will call y; so that the following equation must always obtain, namely,

$$\frac{34(x - p)}{x} + a - x = 34 - y.$$

The value of x deduced from this equation is ambiguous, being thus expressed;

$$x = \tfrac{1}{2}\,a + \tfrac{1}{2}\,y \pm \sqrt{\left((\tfrac{1}{2}\,a + \tfrac{1}{2}\,y)^2 - 34\,p\right)}.$$

Now, when the water ceases to rise, y vanishes, and the equation becomes $x = \frac{1}{2}\,a \pm \sqrt{(\frac{1}{4}\,a^2 - 34\,p)}$; of which the two values are real, so long as $\frac{1}{4}\,a^2$ is greater than $34\,p$. Hence we conclude, that *when one fourth of the square of the greatest height of the piston above the surface of the water in the reservoir is greater than 34 times the play of the piston, there are always two points in the sucking pump where the water may stop in its motion;* and the pump must be reputed bad when the lowest point to which the piston can be brought is found between these two points.

But if $34\,p$ be greater than $\frac{1}{4}\,a^2$, the two values of x, when y is supposed $= 0$, become imaginary; so that in a pump so constructed it is impossible that y should vanish; that is, the pressure of the exterior air always prevails, and the water is not arrested in its passage. Hence we conclude, secondly, that *in order that the sucking pump may infallibly produce its effect, the square of half the greatest elevation of the piston above the water in the reservoir must always be less than 34 times the play of the piston.*

516. This general rule may also be easily deduced geometrically; suppose the valve E to be placed at the surface RS of the water, the tube to be of a uniform bore, and YS to be the height of a column of water whose pressure is equal to that of the atmosphere; that is, $YS = 34$ feet. Let the water be raised by working to N; then the weight of the column of water SN, together with the elasticity of the air above it, exactly balances the pressure of the atmosphere YS. But the elasticity of the air in the space OM, (QO being the highest and CD the lowest situation of the piston,) is proportional to $YS \cdot \frac{DN}{ON}$; and, consequently, in the case where the limit obtains, and the water rises no further, we shall have $YS = NS + YS \cdot \frac{DN}{ON}$. Transposing NS,

Fig 245.

we have

$$YS - NS \text{ or } YN = YS \cdot \frac{DN}{ON};$$

whence

$$ON : DN :: YS : YN;$$

or,

$$ON - DN \text{ or } DO : ON :: YS - YN \text{ or } NS : YS;$$

consequently

$$DO \cdot YS = ON \cdot NS.$$

Hence we see, that if OS, the distance of the piston in its highest position from the water, and DO the length of the half-stroke, or the play of the piston, be given, there is a certain determinate height, as SN, to which the water can be raised by the difference of the pressures of the exterior and interior air; for YS is to be considered as a constant quantity, and, of course, when DO is given, $ON \cdot NS$ is given likewise. To ensure, therefore, the delivery of water by the pump, the stroke must be such that the rectangle $DO \cdot YS$ shall be greater than any rectangle that can be made of the parts of OS; that is, greater than the square of $\frac{1}{2}\, OS$, by a well-known theorem.

Hence we deduce a practical maxim of the same import as the preceding, which is, *that no sucking pump can raise water effectually, unless the play of the piston in feet be greater than the square of the greatest height of the piston, divided by* 136.

517. Resuming the equation

$$\frac{34\,(x-p)}{x} + a - x = 34 - y,$$

and finding thence the value of y, we have

$$y = \frac{x^2 - a\,x + 34\,p}{x}.$$

Now let AB represent the greatest height of the piston above the surface of the water in the reservoir, and AD the play of the pis- Fig. 248, 249.

ton; suppose the different portions AP of the line AB to represent the successive values of x, and lay down upon the perpendiculars PM the values of y which correspond to these assumed values of x; so shall we have a curve MMC, which, while $\frac{1}{4}\, a^2$ is greater
Fig.248. than $34\, p$, will cut AB in two points I and I', in such a manner that the ordinates PM will lie on different sides of AB; the ordinates which are on the right AB showing the positive values of y, and those which are on the left AB the negative values. We see, therefore, that so long as $\frac{1}{4}\, a^2$ is greater than $34\, p$, the pressure of the exterior air is strongest, until the water has attained the height BI'. At this point I' it will stop (the motion acquired being left out of consideration,) because the value of y is $= 0$. But if the water by the motion it has acquired continues to rise till it reaches some point between I' and I, it will not stop there, but will descend, if the valve does not oppose its descending motion; because the value of y, being there negative, indicates that the pressure of the exterior air is weaker than the united pressures of the water and the interior air. If the water reaches the point I, it will stop there, for the same reason that it would at the point I'; but if it rises above I, there is then no reason to fear that it will descend; for all the ordinates PM between I and A being positive, show that in that portion of the pump the pressure of the exterior air exceeds the combined efforts of the interior air and water.

518. When, on the contrary, the value of $\frac{1}{4}\, a^2$ is less than that
Fig.249. of $34\, p$, the curve will not intersect the axis AB; all the ordinates are positive, and consequently the pressure of the exterior air is always the strongest. This confirms and illustrates what has been laid down in article 515.

If the sucking pump were to be placed so high above the usual surface of the earth (as at the top of a high mountain), or so low beneath it (as in a deep mine), that the pressure of the atmosphere would be sensibly different from the assumed mean pressure equivalent to 34 feet of water, we must then in all the preceding investigation change the co-efficient 34 to that which would express the height in feet of the corresponding column of water. And these equivalent columns may always be ascertained by means of the height of the mercurial column in the barom-

eter; the proportion being this; as 30 inches, the mean altitude of the mercurial column, is to 34 feet, the mean height of the column of water, so is any other mercurial column in inches to its corresponding column of water in feet.

519. In the preceding calculation the pump has been supposed of a uniform bore throughout; when this is not the case the solution is rendered somewhat more complex, but not difficult. To calculate the effort of the interior air when the water has not reached the body of the pump, having only attained the height *HN*, for example, we must use this proportion; as the space Fig.245.
QOVNMI : the space *CDVNMI* : : 34 feet : a fourth term, which being added to the weight of the column of water whose height is *NH*, ought again to be equal to 34 — y, as before. Besides, when the sucking pipe *FG* is of a smaller diameter than the body of the pump, if the conditions which we have before specified obtain, the pump cannot fail to produce the proper effect; for the air is dilated with more facility in this latter case than when the whole is of the same diameter. We need only add on this point, that if the length of the stroke in a uniform pump, which is requisite to render the machine effectual, be greater than can conveniently be made, it may be diminished *by contracting the diameter of the sucking pipe in the subduplicate ratio of the diminution of the length of the stroke.*

520. As to the effort of which the power ought to be capable to sustain the water at a determinate height *YH*, it will be measured according to what we have said respecting the lifting pump by the weight of a column of water whose base is equal to *CD*, and height that of *XY* above *RS*. Here, too, we leave out of consideration the friction and the weight of the piston.

521. The velocity of the water flowing from the sucking pipe into the barrel should be equal to the velocity with which the piston moves. For if it be greater, less work will be done than the pump is competent to effect; and if it be less, a vacuum will be produced below the piston, which will therefore be moved upwards with great difficulty.

Of the Syphon.

Fig. 250. 522. If we introduce into a vessel of water or other liquid, the shorter branch of a recurved tube DEF, called a *syphon*, and exhaust the air from this tube by the mouth or otherwise, the water will rise in the tube and flow out at F, until the surface of the fluid in the vessel shall have descended to the opening D of the syphon.

The reason of this phenomenon is, that when the contained air is withdrawn, the pressure of the interior air upon the surface AB causes the fluid to rise into the syphon and to flow through the branch EF. And although when the flowing has commenced, the air presses the fluid at the point F with a force very nearly equal to that which is exerted upon the surface of the water in the vessel, still a transverse lamina at F is urged downward by the entire column of water IF; this column must therefore descend; but in descending it tends to form a vacuum at I, which cannot fail of being filled by the action, always exerted, of the incumbent air upon the surface of the fluid in the vessel.

It will hence be seen that during the discharge through the syphon the air acts only with an effort proportional to the difference of level IF between F and the surface of the water in the vessel; so that the flowing will be so much the more rapid according to the difference of the two branches of the syphon; thus if F and D were on a level, no motion of the fluid would take place. We say generally, that the branch EF must be longer than the branch ED; but in using this language it must be understood that the vertical height of E above F must be greater than that of E above D. The absolute height is not concerned, DE may be rendered much longer than DF by being made crooked; so long as the point D is higher than F, the fluid will pass until it arrives at D provided the height of E above D does not exceed 34 feet.

Of the Steam-Engine.

523. THE whole theory of the *steam engine* is founded upon two principles, the developement of the elastic force of aqueous vapor by heat, and the sudden precipitation of this vapor by cooling. On account of the extensive uses of this machine in the arts, we shall here treat of it at some length.

Although it is generally sufficient in mechanics to create any one force or motion, in order to be able thence to deduce all sorts of motions, yet for the sake of distinctness we shall suppose that it is proposed, in the first place, to draw water from a mine by means of a sucking pump $T' T'$. Here the point in question is Fig. 251.
to raise the piston P'. For this purpose we attach the piston rod to a chain applied to one of the extremities A' of a bent lever moving about its centre C. It is evident, that if we attach to the opposite arm of the lever a similar chain represented by AD, we shall only have to pull this chain, in order to raise the piston P', and draw the water into the body of the pump by the external pressure of the atmosphere. This being done, the valves placed at the bottom of the pump will close; and the apparatus being left to itself, if we suppose the weight of the piston P', together with that of the frame which supports it, to exceed the total weight of AD and P, it is evident that the piston P' will descend into the water by its own weight and cause the water to raise the valve opening through its centre; and having reached the bottom of the body of the pump, will separate this water entirely from the water below. Then by pulling anew the chain AD, we shall raise this water with the piston, and at the same time draw more water into the body of the pump; after which the piston will descend by its own weight in the same manner as before, and so on indefinitely. It remains then to give the requisite motion to the chain AD. For this purpose we attach its lower extremity D to another piston P, moving like the first in the body of the pump TT' likewise cylindrical; but suppose the bottom of

the body of the pump, instead of being immersed in water, to communicate with an air pump by means of which the air in it can be exhausted. It is manifest that after the exhaustion the pressure of the atmosphere upon the upper surface of the piston P will tend to make it descend; and will, in fact, make it descend, if the whole effect of this pressure exceed the weight of the piston P', together with that of the column of water to be raised. Now the piston P, having descended to the bottom of the body of the pump, let the air be admitted below; then the pressure on the two surfaces will be equal; and the excess of the weight of the piston P' beginning to act, P will rise in the bore of the pump; after which, if we make a new exhaustion under P, we shall cause P to descend and P' to rise, and we can repeat these motions at pleasure.

But it will be seen that the air pump could hardly be employed upon so large a scale. To supply its place we introduce steam into the body of the pump TT. The simplest method of doing this, and which, (although it has not hitherto been most generally employed,) is nevertheless attended with some pecu-
Fig. 252. liar advantages, is the following. Under the body of the pump TT, let there be placed a boiler FF, filled in part with boiling water, the steam of which being equal or a very little superior in elasticity to the pressure of the atmosphere, may be introduced at pleasure into the cylinder TT by turning a stop-cock R, which opens a communication between the pump and the boiler. Let there be likewise at the bottom of the cylinder a small lateral passage VS, shut by a valve S opening outwards. Now the piston P being forced to the bottom or nearly to the bottom of the cylinder TT, and the space below being filled with air, turn the stop-cock R which communicates with the boiler. The steam will rush into the cylinder, and by its impulse, together with its elastic force, will in part expel the air remaining in the cylinder by forcing it to open the valve S. In this operation a great quantity of steam is suddenly condensed by the cold surface of the cylinder TT and that of the piston P, and being reduced to water, is made to pass out through a descending tube EGS', recurved at the lower end and terminated by a valve S', opening outwards. This condensation, and consequent loss of steam produced by

cooling, will continue until the piston and the portion of the cylinder situated between it and the boiler, are brought to the temperature of the steam itself. Then, the steam preserving its elastic form under the piston P, counterbalances entirely or partly the pressure of the atmosphere upon its upper surface. The excess of weight in P', acting therefore without obstacle, causes $A'P'$ to descend and AP to ascend, which tends to produce in the cylinder a vacuum into which steam, rising from the boiler, continues to enter, until the piston P, having reached its highest point, the cylinder is completely filled with steam. Having obtained this limit, the steam opens the valve S and escapes, at first slowly and in the form of a cloud, on account of its mixing with the air and drops of water. According as the air is expelled this current becomes gradually stronger and more transparent. When the operator perceives that this point is attained, he turns back the stop-cock, and then the whole cavity of the pump remains filled with pure steam which only wants to be condensed by sudden cooling in order to leave a vacuum under the piston P. This condensation is effected by the introduction of cold water which is made to descend from an elevated reservoir Z through the tube $ZR'I$, closed at R' by a stop-cock called the *injection stop-cock.* Upon turning this, the cold water thrown into the cylinder TT, precipitates entirely or partly the steam contained there and it flows out through the tube EGS' with the water which results from this condensation ; then a vacuum being left under the piston P, the pressure of the atmosphere causes it to descend. This piston is again raised by the introduction of fresh steam ; for if, as we have supposed, the water in the boiler is kept in a state of ebullition, the steam has an elastic force at least equal to that of the air, and consequently its introduction under the piston P is sufficient to counteract the pressure of the atmosphere ; so that the excess of weight in P' will raise P as before. But on the other hand, the steam, if heated too much, may by its elastic force cause the boiler to burst. To guard against this, we adapt to the top of the boiler a *safety valve* S'', which opens outwards with a known effort. When the elastic force of the steam is equal or inferior to that of the external air, the valve remains closed ; but when it becomes equal to that of the atmosphere and the resistance of the valve together, the steam escapes and

no explosion is to be feared. Still, however, it is necessary that the boiler should be made stronger than this limit of resistance supposes. For when the steam rushes into the cold cylinder and is condensed there, this precipitation is so rapid that the new steam formed at the same time in the boiler is not always sufficiently instantaneous to supply its place. For a moment a vacuum is left in the boiler, and the pressure of the external air being no longer counterbalanced, the boiler may burst inward if its sides are not sufficiently thick. This accident sometimes happens, but it may always be prevented by means of a second *safety valve*, which opens inward whenever the external pressure becomes too great.

From this explanation it would seem that when the engine was once in operation nothing would remain in the piston or body of the pump but pure steam or a vacuum. But it must be remarked that the injected water has also some air combined with it which escapes into the body of the pump; since it is found there in a highly rarified state, being heated to a considerable degree by the great quantity of heat disengaged during the condensation. Happily this air, being in small quantity, and contained in a small space, is easily expelled through the valve S by the first effort of the steam introduced into the cylinder.

524. The apparatus which we have described is not precisely the one first invented. It appears that the original attempt was simply to employ the force of steam as a moving power. But the more ingenious discovery of the method of condensing the steam by cooling was not made until 1696; and the English attribute it to Capt. Savary, who published an account of it in a treatise entitled *The Miner's Friend*. The mode of applying this principle was still very imperfect. In 1705, Newcomen, another Englishman, gave it the form which we have described, in which, under the name of the *atmospherical engine*, it was a long time not unprofitably employed.

Nevertheless, with the progress we have made in mechanics and the natural sciences, it is easy to perceive that this engine had many theoretical defects. It was a great imperfection that it required an intelligent person to watch it for the purpose of turning the stop-cocks to introduce water and steam every time the

piston reached its limit. A good machine ought always to tend itself by means of the first moving power, without any foreign aid. Another great inconvenience was the introduction of steam into a cold cylinder; since a great loss was thereby occasioned, and was repeated at each stroke of the piston, the cylinder being continually cooled by the injected water necessary for the precipitation. But these defects, which in the present state of the sciences we are so prompt to observe, could not at first have been so easily detected. They were perceived and corrected in 1764, by Watt, the disciple and friend of Black. Being then at Glasgow, where he was employed as a mathematical instrument maker, he was directed to repair a small model of Newcomen's engine, which belonged to the University in that city. In the course of his attempts to make its operation satisfactory, he observed that it consumed more coal in proportion to its size than the large engines. Being curious to ascertain the cause of this difference, and wishing to remedy so great a defect, Watt made numerous experiments for the purpose of determining what substance is the most suitable for the cylinders; and what are the most proper means of creating a perfect vacuum; what is the temperature to which water rises in boiling, under different pressures; and what the quantity of water necessary to produce a given volume of steam, under the ordinary pressure of the atmosphere. He determined also the precise quantity of coal necessary to convert a known weight of water into vapor, and the quantity of cold water required to precipitate a given weight of steam. These several points being once exactly ascertained, he was led to perceive the defects of Newcomen's engine, and to assign the cause of these defects. He saw that the steam could not be condensed so as to produce any thing like a vacuum, unless the cylinder and the water it contained, as well as that injected and that arising from the condensed steam, were cooled down at least to the temperature of about 33°, and that at a higher temperature the steam had still an elasticity strong enough to oppose a very perceptible resistance to the weight of the atmosphere. On the other hand, when it is proposed to attain more perfect degrees of exhaustion, the requisite quantity of injected water is augmented in a very rapid proportion; and hence results a great loss of steam, when the cylinder is again filled. These facts led Watt to conclude, that, in order to effect

the most complete vacuum possible, with the least expense of steam, it was necessary that the cylinder should be kept constantly as hot as the steam itself, and that the injection of cold water should take place in a separate vessel which he called the *condenser*, and whose communication with the cylinder was suddenly opened at the moment of the injection. Indeed after what is now known respecting the equilibrium of fluids, it is manifest that, if the air be exhausted from the condenser, the steam from the cylinder will enter it, on account of its own elasticity, the instant a communication is opened; and an injection of water made at this instant, will precipitate not only the steam actually in the condenser but also on the same principle, all the steam contained in the cylinder, which, rushing into the vacuum, continually formed by precipitation in the condenser, is converted almost instantaneously into water. It only remains, then, to remove this water and disengage the air, in order to preserve always a vacuum in the condenser. Watt constructed a pump in such a manner as to be moved by the engine itself and which played continually in a tube void of air, the lower part being immersed in the water of the condenser. Finally, the condition of keeping the cylinder hot could not be fulfilled while there was a free admission of atmospheric air to the interior of its upper surface, which in the apparatus of Newcomen, caused the piston to descend; especially, since in order to prevent the passage of the steam between the cylinder and piston, the latter was ordinarily covered with a stratum of cold water which kept the interior of the cylinder wet. Watt conceived the bold and ingenious idea of dispensing entirely with the pressure of the atmosphere, and making the piston descend by the force of steam alone, by introducing it alternately above and below, and causing at the same time a vacuum in each case in the manner already described. Then he enclosed the rod of his piston in a collar of leather to prevent all access of air to the interior of the cylinder; and employing steam of an elasticity equal or even a little superior to the pressure of the atmosphere, he obtained alternately above and below the piston a force equal, or a little superior, to the atmospheric pressure. He was then able, by substituting stiff rods for the chains AP, $A'P'$, to produce a force in both directions; whereas, in Newcomen's engine, the time during which the piston was ascending in the cyl-

inder, was entirely lost so far as the steam was concerned, since it was raised simply by the excess of weight on the other arm of the large lever. Here was a saving both of time and expense; for the piston was accelerated each way by steam, and the quantity of fuel employed to keep it hot, during its ascent, was not wasted. Watt took care, moreover, to surround the cylinder with a case of wood, or some other substance which is a non-conductor of heat; into the interior of which, he also occasionally introduced the steam as a means of keeping it warm. He likewise used so much economy in the construction of the different parts of his engine, that he succeeded in saving two thirds of the steam employed by Newcomen. The steam-engine, thus improved, is represented in figure 253, the explanation of which will now be easily understood. *FD* is the boiler, in which the water is converted into steam by the heat of the furnace below. This boiler is sometimes made of copper, but more frequently of iron. Its bottom is concave and the flame curls around it. Towards the top, it has a safety valve S'' fitted to resist a greater or less effort before opening, according to the degree of elastic force to be employed. That the conversion of water into steam may be constant, it is necessary that the water in the boiler should be kept always at the same level, and consequently that it should be supplied as fast as it is evaporized. This is effected by a tube $v\ v$, which supplies the boiler from a small reservoir z, filled with water, already heated, which the pump $t\ t$ takes from the condenser and forces into the lateral pipe $t'\ t'$. But in order to introduce this into the boiler only when it becomes necessary, the upper orifice of the tube $v\ v$ is closed by a stopper, which is raised or lowered by means of a small lever $a\ b$; and at the other arm b of the lever hangs a wire $b\ m$, drawn downwards by a weight m, which is so adjusted in the boiler as to keep on a level with the upper surface of the water. Then if the water falls below this level, the weight m, which it supports in part, will descend with it; the lever turning will raise the stopper, and suffer the water to pass into the boiler; but as soon as the level is reëstablished, the lever $a\ b$ will again become horizontal, and return the stopper to its place. From the top of the boiler proceeds the steam tube VV, which conveys the steam to the top of the cylinder TT, by the valve S, and to the bottom, by the

valve S'. The communication between S and S' is effected by means of a curved pipe, whose plane of curvature is perpendicular to the plane of projection in figure 253. One part only of this pipe is represented in this figure; in order that the two valves S'', S''', may be exhibited, of which we are presently to speak; but the whole may be seen in figure 254, where it is presented in profile. The valves S'', S''', are those by which the steam of the cylinder is brought into communication with the condenser, on both sides of the piston; and they are opened and closed at proper times by the engine itself, with the aid of two small projections, 1, 2, attached to the rod $t\ t$ of the pump which serves to exhaust the condenser C. This opening is effected a little before the piston has completed its vertical motion, and the communication is then established between the two surfaces, in order that the equality of pressure thence resulting may weaken the effort which would be made if it were exerted on one side only, and prevent the sudden jar which would follow from the piston's striking with its full force against the bottom of the cylinder.

These are the principal conditions relating to the action of the steam, but there are others which relate to the manner of directing this action. Indeed, a bare inspection of the figure will show that the rod of the great piston and that of the pump which exhausts the condenser, being both inflexible, cannot be attached immediately to the large lever AB; for each point of this lever, describing an arc of a circle about its centre of rotation, would tend to change the point of attachment from a vertical direction, and this effort would break the engine. It is on this account, that, in the engine of Newcomen, where the piston acted only in its descent, its communication with the large lever was made by a chain applied to the arc of a circle. But in the engine under consideration, the inflexibility of the rods requires some other mode of communication. Mr. Watt effected this by means of a particular assemblage of metallic bars, moving upon one another, and which compensate by their action, for the want of perfect verticality in the motion of the large lever. The figure represents also, several other useful appendages to the engine, such as flies to regulate its motion, and wheels to transmit it. There is also a very essential part designated by G, and called the *governor*. It consists of a vertical rod which is kept contin-

ually in rotation by the engine itself, and which carries at its summit a parallelogram formed of metallic plates, turning freely one upon the other, in such a manner that this parallelogram may open more or less in a horizontal direction, according as the plates diverge more or less from the axis. This divergence is produced by the centrifugal force exerted by the axis in turning, under the influence of the engine, with greater or less rapidity; which causes the upper vertex of the parallelogram to be depressed when the engine moves more rapidly, and elevated when it moves more moderately. In order to give greater force to this ascending and descending motion, the extremities of the plates are loaded with spheres of solid metal, and exert their power upon a lever, whose other branch communicates with a plate placed transversely in the passage through which the steam passes from the boiler to the cylinder; so that, when the engine works too slowly, the plate turns in such a manner as to give a freer passage to the steam; and, on the other hand, if the engine moves too rapidly, the plate takes a position more nearly in a transverse direction, and diminishes the passage through which the steam has to pass. Thus the engine is made to govern and regulate itself, in such a manner as to preserve in its motions that uniformity which its purposes require. There are many other details which would furnish matter for curious speculation; but these details, belonging to the mechanism, must be omitted here, to make room for some other particulars, no less important, relating to the principles of the machine.

525. The most essential of these is the determination of the temperature, at which it is most advantageous to employ the steam. In fact, the higher the temperature is, the greater will be its elastic force, and consequently the greater will be its effect upon the surface of the piston, the vacuum being always on the other side. From the experiments of Southern, Clement, Desormes, and Despretz, it has been found that the total quantity of heat necessary to change the same mass from water to a state of vapor, is very nearly if not quite the same for all temperatures. According to this principle, then, it will not be necessary to consume more fuel in order to form a given weight of steam of a higher temperature and more elastic, than is required

for the same weight at a lower temperature and less elastic. But while the steam is raised to a higher temperature, it becomes also more dense; so that considerably more weight is necessary to fill the same cylinder, than when its temperature is lower. This second circumstance, therefore, requires in the same machine, a greater quantity of fuel, according as the temperature of the steam is increased; so that it only remains to ascertain whether this increase of fuel is, or is not, compensated by a corresponding increase of elastic force. Now this point is easily decided, if we reflect that the vapor of water and that of other liquids, so long as they exist in a state of vapor, are subject to the same physical laws of compression and dilatation, as the permanent gases. Accordingly, let w represent the weight of a cubic inch of aqueous vapor, such as it would be, if it were capable of being reduced to the temperature of melting ice, and under a barometric pressure of 29,92 inches; * and let w' represent the weight of an equal volume of the same vapor, as it would really exist at another temperature t, and with the elastic force F, equal or inferior to the maximum density belonging to this temperature. If we put $t' = t - 32$, or the number of degees distant from freezing, according to the laws of
470. the dilatation of the permanent gases, we shall have

$$w' = \frac{w\,F}{29{,}92\,(1 + t' \,.\, 0{,}00208)}.$$

All the experiments of philosophers upon aqueous vapor, and those of Despretz upon the vapor of several other liquids, show that this formula is really applicable to them; so that we may safely employ it for the purpose under consideration. Now calling c' the quantity of heat necessary to convert 15,5 grains Troy † of water, in a liquid state, at 32°, into vapor, having the temperature t'; $c'\,w'$ will be the absolute quantity of heat necessary to form the weight w' of such a vapor; and consequently
282. if n be the number of cubic inches contained in the cylinder to be filled, the whole quantity of heat necessary for this purpose, will be

$$\frac{c'\,n\,w\,F}{29{,}92\,(1 + t' \,.\, 0{,}00208)}.$$

* Or, more accurately, 15,444 grains, or one gramme.

† Equal to 0,76 of a metre.

Having thus brought into an equation all the different elements from which this expenditure of heat results, we are able to analyze the several effects produced by them. In the first place, according to the experiments of Clement, Southern, and Despretz, before mentioned, c' is sensibly constant at all temperatures at which we have yet been able to make observations, and accordingly the expenditures will be always the same. As to the elastic force F, we know that it augments according as the temperature is raised. But it will be seen by the formula that the density w', and consequently the expenditure of heat, increases in proportion to F. Consequently if we take away the factor $(1 + t' \,.\, 0{,}00208)$, arising from the dilatation produced by the increase of temperature, the expenditure necessary to produce the force F will be exactly proportional to this force, and we shall neither gain nor lose any thing by giving it a greater or less energy. But the influence of the factor $(1 + t' \,.\, 0{,}00208,)$ which increases as the temperature is raised, diminishes the expenditure in question, in proportion as the temperature becomes higher; for if we operate, for instance, at 212°, this factor becomes 1,375; and if at 320°, it becomes 1,600; so that the relative expenditure in this last case, compared with that in the first, with the same elastic force, is nearly in the ratio of 1375 to 1600, or nearly of 6 to 7. Such then is the saving of heat that may be made in the formation of steam in the extremes of temperature at which we have as yet operated. Accordingly if it were true that high-pressure engines have, over those of the ordinary kind, an advantage as great as has been alleged, it must be sought in something else beside the saving of fuel. But in order to judge correctly of these engines it is necessary to take into consideration another element, namely, the ulterior developement of elastic power, which the steam thus formed is capable of furnishing, when, after having been employed with its primitive energy in the first cylinder, it is made to pass into another larger cylinder where it dilates, undergoing a reduction of temperature at the same time, in such a manner as always to fill the space into which it is received, until at length it is condensed into water, when it no longer has an elastic force, either equal or inferior to the pressure of the atmosphere. It is manifest that this ulterior developement of force must, in order that the whole effect may be appreciated, be added to the mechanical power at the

commencement; and it is no less evident that it offers a peculiar advantage to engines in which the steam, before being condensed, is employed at a high temperature. Nevertheless, the numerous experiments which have been made within a few years upon engines of this kind, some of which have been accompanied with careful measurements, have not seemed to confirm, so much as might have been expected, the favorable view we have given above; and if they have taught us a real saving of fuel, when considered with reference to the force actually developed, this saving does not appear to exceed the narrow limits just assigned to the effect of the heat of dilatation. So small an advantage will be far from compensating for all the additional precautions required in such engines, the dangers incurred, and the numerous causes of waste to which they are liable. It is necessary, in the first place, to make the boilers, cylinder, &c., very strong, that they may resist the expansive force exerted by the steam. It is likewise necessary to give greater perfection to the pistons, and to apply more frequently some lubricating substance to preserve the contact. Repairs are in consequence often required, and the value of the engine is sensibly diminished on this account.

Attempts have been made to remove this great inconvenience by an expedient formerly employed to perfect the first invention of Savary. It is this; the piston, instead of being in immediate contact with the aqueous vapor, which melts and dissolves the grease with which it is impregnated, receives its motion through the intervention of a column of oil, or some other unctuous substance not easily evaporated, on which the steam is made to act by pressure. For this purpose the cylinder in which the piston plays is enclosed in a larger cylinder with which it communicates, and which contains the oil. The oil rising and falling continually in the interior cylinder keeps it always lubricated. But although this ingenious arrangement may be sometimes adopted with advantage, it could not be used at high temperatures; for, according to the judicious observation of Mr. Watt, the oil would be decomposed by the dissolving power of the steam.

It has likewise been proposed to construct high-pressure engines having a very great power with very little bulk, by employing small boilers, made so strong as to resist the most

powerful pressure, while they admit of being heated, as it were, red hot; whereby steam would be obtained of an excessively high temperature, and which, developing itself by its expansive force in the great cylinder, would possess even after its dilatation an elastic force sufficiently energetic. A similar arrangement was likewise attempted formerly, in order to furnish steam for Savary's engine; but it does not appear to have been found profitable enough to be continued in use; and after the calculations we have made on the expenditure of heat employed in the formation of steam at every temperature, it seems improbable that a sufficient saving can be made in such engines to compensate for their inconveniences. Advantages of a different kind might be realized, if we could succeed in employing, without loss, some liquid different from water, having a much greater elastic force at the same temperature.

One inevitable consequence of employing high temperatures is, that the loss of heat by radiation is much greater, and this makes an important item in calculating the results. To form an idea of the diminution of effect arising from these different circumstances, we are to remember that according to Lavoisier and Laplace, 1 gramme or 15,444 grains Troy of charcoal, developes, in burning, 13038 degrees of heat by Fahrenheit's scale, or about 92° on Wedgewood's. Now 15,444 grains of water at the temperature of 212°, by being converted into steam absorb 1020,6° by Fahrenheit; then 15,444 grains of charcoal would reduce to steam 200 grains of water, on the supposition that no heat is lost, and that the water is already brought to the temperature of 212°. But after a great number of experiments made upon the most perfect engines and the best constructed furnaces, Mr. Clement found that 1544,4 grains of charcoal does not produce more than 6 or 7 times as much steam, and 1544,4 grains of the best fossil coal never produces more than 6 times as much steam; whence it will be seen that nearly half the heat is lost by radiation, and the conducting power of the boiler and the surrounding bodies. The loss is without doubt still more considerable in engines of a high pressure.

526. When we know the elastic force of the steam introduced under the surface of the piston, it is easy to estimate the whole pressure resulting from it; but in this estimate it is necessary to

take account of the tension of the steam which remains on the other side when the vacuum is not perfect. After all, this estimate fails of giving the primitive energy of the power employed, and much less the part which remains to be disposed of after all the friction is overcome and the different parts of the machine are put in motion. This useful part of the force can be measured *a posteriori* by the effect which the whole engine produces. Ordinarily we compare the effective power of an engine with that of a certain number of horses of a medium strength, and the force of the engine is estimated accordingly. By a great number of experiments of this kind Watt and Bolton supposed that a horse of ordinary strength, working 8 hours a day, would raise 3200 lb. avoirdupois one foot per hour. Smeaton made the estimate 2300, and Clement about 1300. If, therefore, we divide the number of pounds that a steam-engine will raise one foot (or, which is the same thing, the product of the number of pounds into the number of feet elevation), by the number representing a horse-power, the quotient will be the number of horses to which the engine is equivalent. There are engines of the power of 20, 30, &c., horses. The most powerful engine that has yet been constructed is that employed in the mines of Cornwall. It has the power of 1010 horses, and serves to drain by pumps a mine 590 feet deep. It is evident, that the power in question is all that needs to be estimated; for we can apply it to the raising of water, the turning of spindles, or to any other purpose that requires such a force. The transmission of the primitive motion can be effected by mechanical means and instruments of which we have already spoken, and which need not be again described.

NOTES.

I.

On the Measure of Forces.

The name of *living forces* has been applied to the forces of bodies in motion, and that of *dead forces* to those, which, like a simple pressure, suppose no actual motion in the operating cause.

There was, for a considerable time, a difference of opinion among mathematicians in regard to the measure of *living forces*, or the forces of bodies in motion. Some affirmed that these forces ought not to be measured by the product of the mass into the velocity, according to the rule that has been given, but by the product of the mass into the square of the velocity. As this difference in the estimate of forces may be deemed of great importance in mechanics, it will be proper to make a few remarks upon it.

It is altogether unimportant whether we measure the force of bodies in motion by the product of the mass into the velocity simply, or by the product of the mass into the square of the velocity, provided that in the two cases we assign different significations to the word *force*. When we assume as the measure of force, the product of the mass into the square of the velocity, we mean by the term *force* the *number* of obstacles which a moving body is capable of overcoming; it is certain that the number of obstacles which moving bodies of equal masses are capable of overcoming is proportional to the squares of the velocities. For example, if the body A has Fig. 255.
precisely the velocity necessary to close the spring ACB, an equal body M will require only double this velocity to close four springs, each equal to ACB. For, in the first instant, for example, the body M advancing with a velocity double that of A, will close the four springs, considered as one, twice as much as A would close its single spring; each of the four springs then will be closed only half as much as ACB, and consequently will have opposed during this instant a resistance only half as great as that of ACB; all the four

springs, therefore, in being each reduced the same angular quantity as ACB is reduced, will oppose only double the resistance. In the same manner it may be demonstrated that, in the succeeding instants, the resistance opposed by the four springs in being closed each the same angular quantity, as that by which ACB is closed in one instant, is always to that opposed by the single spring ACB, in the same instant, in the ratio of the velocities belonging to the two bodies; therefore a velocity double that required to close one spring is sufficient to close four springs. Thus the number of springs closed, which are 1 and 4, are as the squares of the velocities 1 and 2 necessary to close them.

We see then that the number of obstacles which bodies in motion are capable of overcoming increases as to the squares of the velocities. But by the term *force* ought we to understand the *number* of obstacles? Is it not much more natural to consider it as denoting the *sum* of the resistances opposed by these obstacles? For it is not merely the number, but the value of each obstacle which destroys motion. Now in this case each instantaneous resistance being evidently proportional to the quantity of motion destroyed by it, (on which point all are agreed,) the sum of the resistances will be proportional to the quantity of motion destroyed. If then by *force*, we understand the *sum*, and not merely the *number* of the resistances which a moving body is capable of overcoming, the force is proportional to the quantity of motion. From this principle it has likewise been inferred that the number of resistances overcome are as the squares of the velocities. The question then is in reality nothing more nor less than a question about terms, and reduces itself to finding the meaning of the word *force*. As to this point we are perfectly at liberty; provided we employ that which we take for the measure of force agreeably to the idea which we attach to the term force, we shall always arrive at the same results. We shall, therefore, continue to take for the measure of forces the product of the mass into the velocity; and consequently by the force of a body we understand the sum total of the resistances necessary to exhaust its motion.

II.

On the Compressibility of Water.

The phenomenon of the transmission of sound through water and other liquids had long indicated that they were capable of being compressed. Canton, an English philosopher, clearly detected this

property by observing the volume occupied respectively by oil, water, and mercury, first placed in a vacuum, and afterwards exposed to the pressure of the atmosphere; but the results which he obtained, though exact in themselves, were, however, liable to be affected by the accidental variations of form and temperature to which the apparatus was subject. M. Oersted completely removed these difficulties by plunging the liquid to be compressed, together with the vessel containing it, into another liquid to which the pressure was applied, and through which it was made to pass to the interior liquid without changing the form of the vessel, since it acted equally within and without. M. Oersted found, likewise, that a pressure equal to the weight of the atmosphere produces in pure water a diminution of volume equal to 0,000045 of its original volume. The experiments of Canton gave 0,000044. M. Oersted found, by varying the pressure from $\frac{1}{3}$ of the weight of the atmosphere to 6 atmospheres, a change of volume sensibly proportional to the pressure. Later experiments, made by Mr. Perkins, seem to show that this proportionality continues when the pressure amounts to 2000 atmospheres. Before the water, however, is entirely freed from air, the diminution of volume, produced by the pressure, is at first somewhat greater than the above ratio would indicate.

III.

On the Condensation of Gases into Liquids.

Mr. Faraday enclosed in glass tubes, bent and sealed, different chemical products which were capable of developing gases by their mutual combination. He introduced them into the tubes in such a manner that they remained separate in the different branches of each tube, and were not mixed until the tube had been sealed. All the gas developed in each tube was found to be confined to a fixed volume, to which it could be reduced only by the action of a considerable pressure; this pressure causes it to liquify in the following cases. 1. Sulphurous acid produced by the action of sulphuric acid on mercury. 2. Sulphuretted hydrogen produced by the action of hydrochloric acid on the sulphuret of iron in fragments. 3. Carbonic acid produced by the action of sulphuric acid on carbonate of ammonia. 4. Oxide of chlorine, produced by chlorate of potash and sulphuric acid. 5. Ammonia disengaged from the combination

of this substance with chloride of silver, &c. In each experiment the branch of the tube containing the mixture was warmed, while the other was cooled with moistened paper.

IV.

On the Construction of Valves.

A valve is a kind of lid or cover to a tube or vessel, so contrived as to open one way by the impulse of any fluid against it, and to close, when the motion of the fluid is in the opposite direction, like the clapper of a pair of bellows. In the air-pump this purpose is effected by means of a strip of leather, bladder, or oiled silk, stretched over a small perforation in the piston, and ordinarily in a
507. plate at the bottom of the barrel.

In common water-pumps the valve, or *sucker*, as it is often called, is a thick piece of leather pressed down by a small wooden weight, and turning on the flexible leather as a hinge. It is represented at *E*, figure 245, &c. In the best metallic pumps for raising water the valve consists of a metallic cone with a stem and knob, as represented at *L* in figure 247. The conical part is ground so as to fit accurately the rim of the opening in which it plays, and, unlike other valves, it is rendered tighter by use, and is less likely to be obstructed by foreign substances contained in the water.

Mr. Perkins invented a pump having a square bore, in which the valve consists of two triangular pieces of leather loaded with weights, and turning on a metallic hinge placed diagonally across the bore. Mr. Evans adapted a similar kind of valve to the common pump of a circular bore, the form of the valve being a semi-ellipse. The chief advantage of this construction is, that there is very little obstruction to the motion of the water, and consequently less loss of power, than in the common pump, where the space, left for the passage of the water, bears a less proportion to the whole bore.

V.

On the History and Construction of the Barometer.

The barometer takes its origin from the experiment of Torricelli, who in consequence of the suggestion of Galileo with regard to the ascent of water in pumps, proceeded in 1643 to make experiments with a tube filled with mercury, conjecturing that as this fluid was about thirteen times heavier than water, it would stand at only one thirteenth of the height to which water rises in pumps, or at about thirty inches. He, therefore, filled a glass tube about three feet long with mercury, and upon immersing the open end in a vessel of the same fluid, he found that the mercury descended in the tube, and stood at about twenty-nine and a half Roman inches, and this vertical elevation was preserved, whether the tube was perpendicular or inclined to the horizon, according to the known laws of hydrostatical pressure. This celebrated experiment was repeated and diversified in several ways with tubes filled with other fluids, and the result was the same in all, allowance being made for difference of specific gravity, and thus the weight and pressure of the air were fully established. Such, however, was the force of prejudice that many refused to yield their assent till, at the suggestion of Pascal, the experiment was performed at different heights in the air with such results as left no longer any doubt upon the subject.

Great care is necessary in the construction of the barometer. The tube after being cleansed as perfectly as possible, is to be gradually heated, and to be kept at a pretty high temperature for a considerable time, for the purpose of expelling all moisture that may be found adhering to it. The mercury is then to be introduced; a small quantity only is first poured in by means of a fine funnel and thoroughly boiled in order to free it from air; then another portion is added, and so on till the tube is filled. It is afterward to be carefully inverted, and the open end immersed in a cistern of boiled mercury.

The tube and cistern is enclosed in a metallic or wooden framework, containing the graduations and some necessary appendages. As the mercury rises and falls in the tube by the fluctuations of the atmosphere, its surface varies also in the cistern. But the graduation is intended to mark the exact length of the column, reckoned from this variable surface. If a horizontal section of the tube and cistern have a constant ratio to each other throughout the extent

embraced by these changes, a correction could be readily applied. Suppose, for instance, that a section of the cistern is one hundred times that of the tube, or that their diameters are as 1 to 10, and that the surface of the mercury in the cistern coincides with the point from which the graduations commence when the mercury in the tube stands at thirty inches. The correction would be one hundreth part of the difference from 30 inches, and additive or subtractive, according as this difference was below or above 30.

It is usual, however, in the best barometers to bring the surface of the mercury in the cistern to the point from which the graduations commence by means of a screw V, acting on a flexible piece of leather which forms the bottom of the cistern. We are able to tell when the desired coincidence is effected by means of a mark on a piece of ivory floating on the mercury and sliding over a fixed object having a corresponding mark. Sometimes the cistern is of glass, and the point of commencement of the graduations is marked upon it, or (which is much better) is indicated by the contact of a sharp ivory pin P, inserted in the cap, and descending into the interior of the cistern.

Fig. 232. Fig. 231.

There are various kinds of portable barometers constructed for the purpose of measuring the heights of mountains. The latest and most convenient is represented in figure 232. It is of the syphon form, and was invented by Gay-Lussac. The barometer being filled, the extremity of the shorter branch Y is hermetically sealed. In this state the barometer is inaccessible to the external air, and consequently is incapable of indicating the changes of pressure in the atmosphere ; but to open the communication, we draw out, by means of a blow-pipe, a small portion of the glass near the middle of the shorter branch, on the inside, and form a fine capillary tube, which is sufficient to admit the air but does not allow the mercury to escape, on account of the force with which it repels it in virtue of its capillary action. The difference of level between the two extremities S, N, of the column being observed, it is reversed, and a part of the mercury enters the longer branch CX, and fills it, the rest falls into the shorter branch CY, but cannot escape for the reason above mentioned. It may then be carried in this position, being always open to the air, but not to the mercury.

This barometer may be enclosed in a cane and transported with great ease and safety. A small thermometer is appended, as in other cases, for the purpose of measuring the temperature of the mercury. By contracting the tube near the two ends we prevent all danger of its breaking by any sudden motion in the column of mercury.

By observing regularly the height of the barometer for a considerable period in the same place, we find that it does not remain constantly the same. For some time after the instrument was invented, it was supposed that the mercury stood higher just before rain, and lower during fair weather. Reasons were assigned for this supposed fact. It was said, that when it is about to rain, the air is charged with water, and consequently that the weight of the atmosphere is more considerable; and that, on the contrary, this weight must be less in fair weather, because the air is then relieved from a certain part of the moisture contained in it. Unfortunately for this hypothesis it has been found, more recently, that the quantity of water which the air is capable of containing, increases with the temperature, so that in summer it contains, for the most part, more water than in winter, although there is less fair weather in winter than in summer. It appears also that the vapor of water is lighter than the same volume of air, when the same elastic force is exerted; that is, if we substitute for a cubic foot of air, taken at a certain height in the atmosphere, a cubic foot of aqueous vapor, of the same temperature and elasticity, the vapor will weigh less than the air, and will consequently exert less pressure upon the barometer. We hence draw a conclusion the reverse of that which the first observers of the barometer undertook to maintain, namely, that the rise of the barometer indicates fair weather, and its fall, rain. This is in fact agreeable to observation in ordinary cases. But it must be confessed that the reason now given is but little better than that we have been combating.

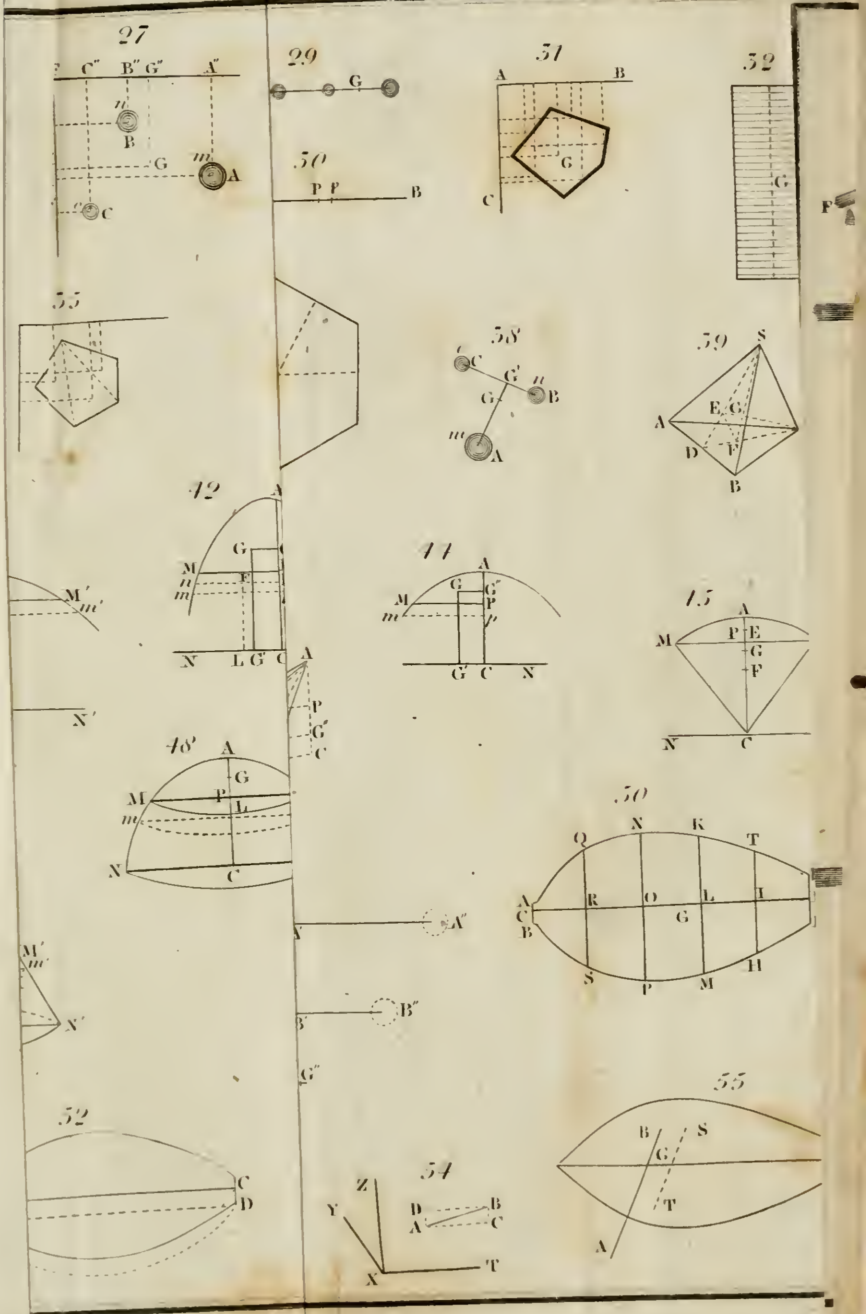
The variations of the barometer are different in different places. They are almost nothing upon the tops of high mountains, and between the tropics; even in the temperate zones they are never very great in calm weather. But the barometer almost always descends rapidly before a violent storm, great changes taking place in a few hours. On this account the instrument is particularly useful at sea.

By comparing observations made at different and remote places, we discover a remarkable correspondence, which shows a simultaneousness in the motions of the atmospheric strata that would hardly have been expected. Still this correspondence is far from being perfect, especially as to the quantity of the change.

By examining a long series of observations made in the same place, we shall perceive, amid all the accidental irregularities, that there is a general tendency, occurring periodically, to rise and fall at certain hours of the day. By a long series of observations, direct-

ed to this point, M. Raymond discovered, that in France the barometer attains its maximum elevation about 9 o'clock A. M., after which it descends till about 4 P. M., when it is at its minimum; from this time it rises till near 11 P. M., when it reaches its maximum again, after which it commences a downward motion till 4 A. M.; and thence it begins to return to the state first mentioned. This march is often deranged in European climates, where the atmosphere is so variable; but under the tropics where the causes which act upon the atmosphere are more constant, the periodical changes are regular, and to such a degree that, according to Humboldt, one may almost predict the hour of the change at any time from a single observation; and, what is very remarkable, these changes, according to the same distinguished philosopher, are not affected by any atmospherical circumstance; neither the wind, nor rain, nor fair weather, nor tempests, disturb the perfect regularity of these oscillations. They are found to be the same in all weathers and at all seasons. For further particulars relative to the construction of the barometer and the theory of its fluctuations, the student is referred to Daniell's *Meteorological Essays.*

THE END.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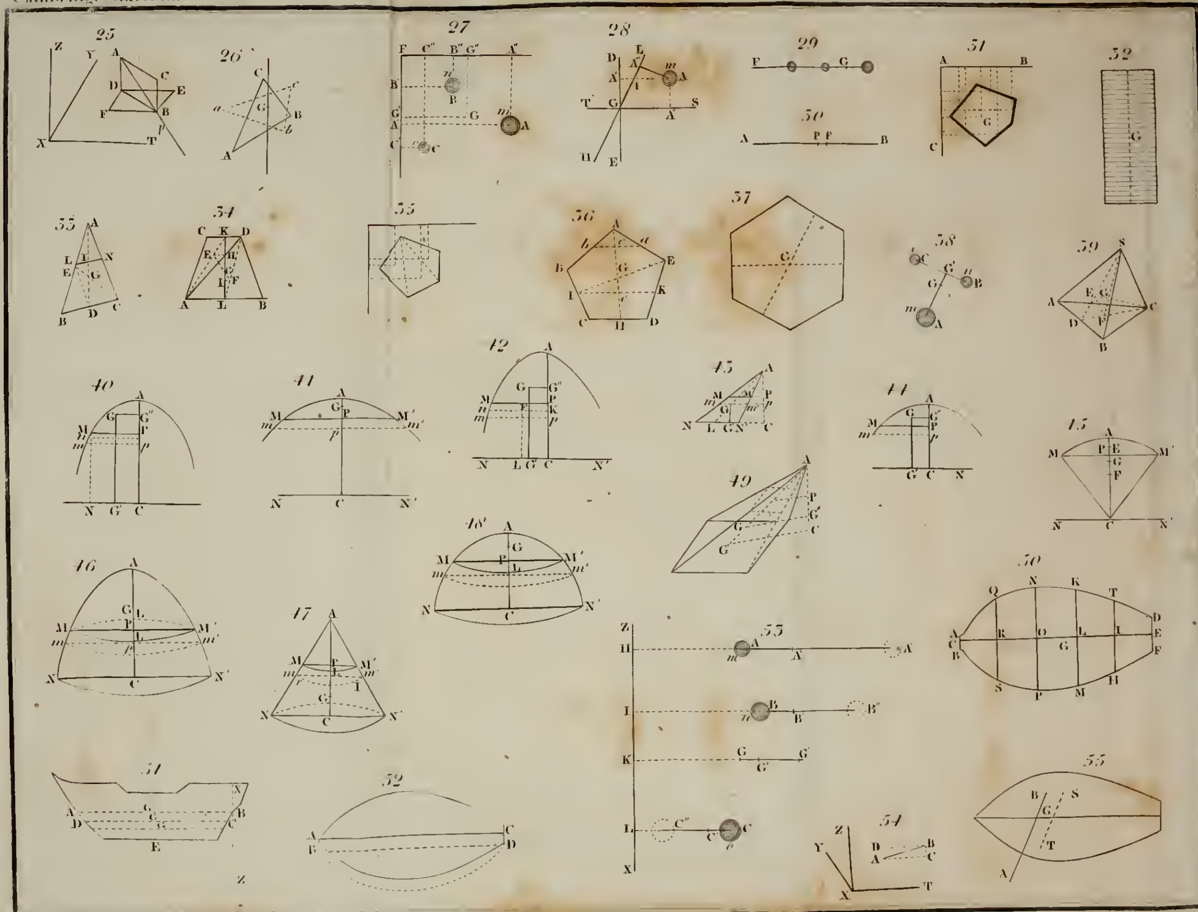
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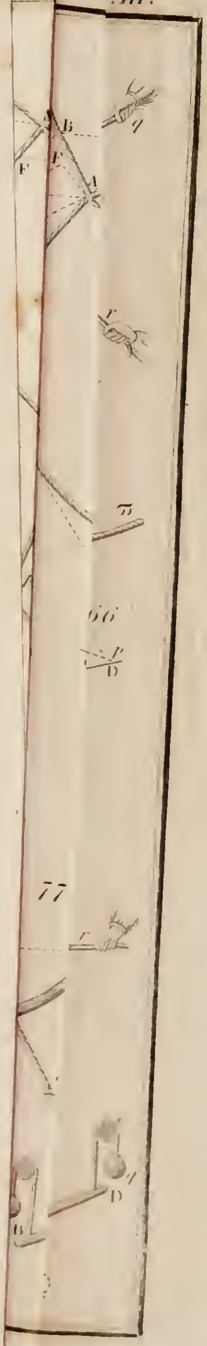

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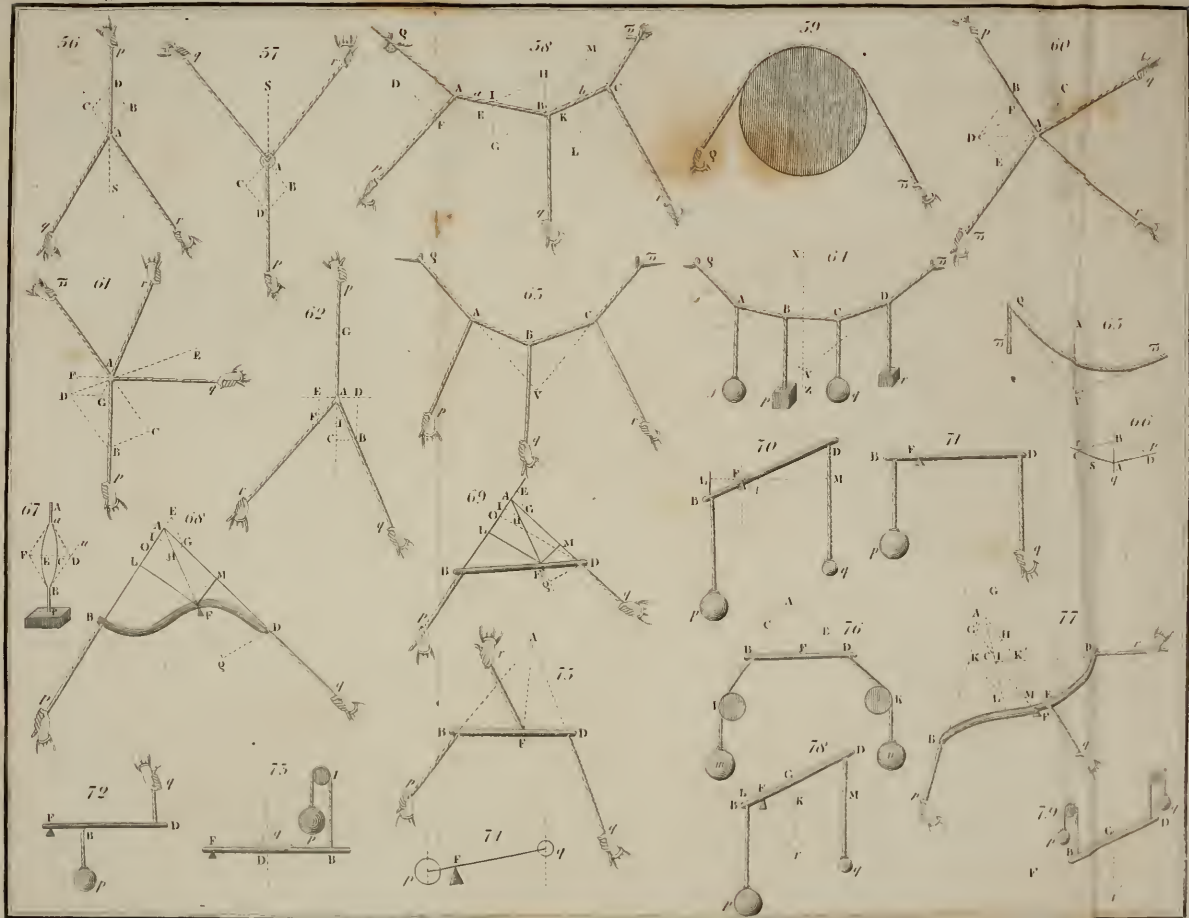

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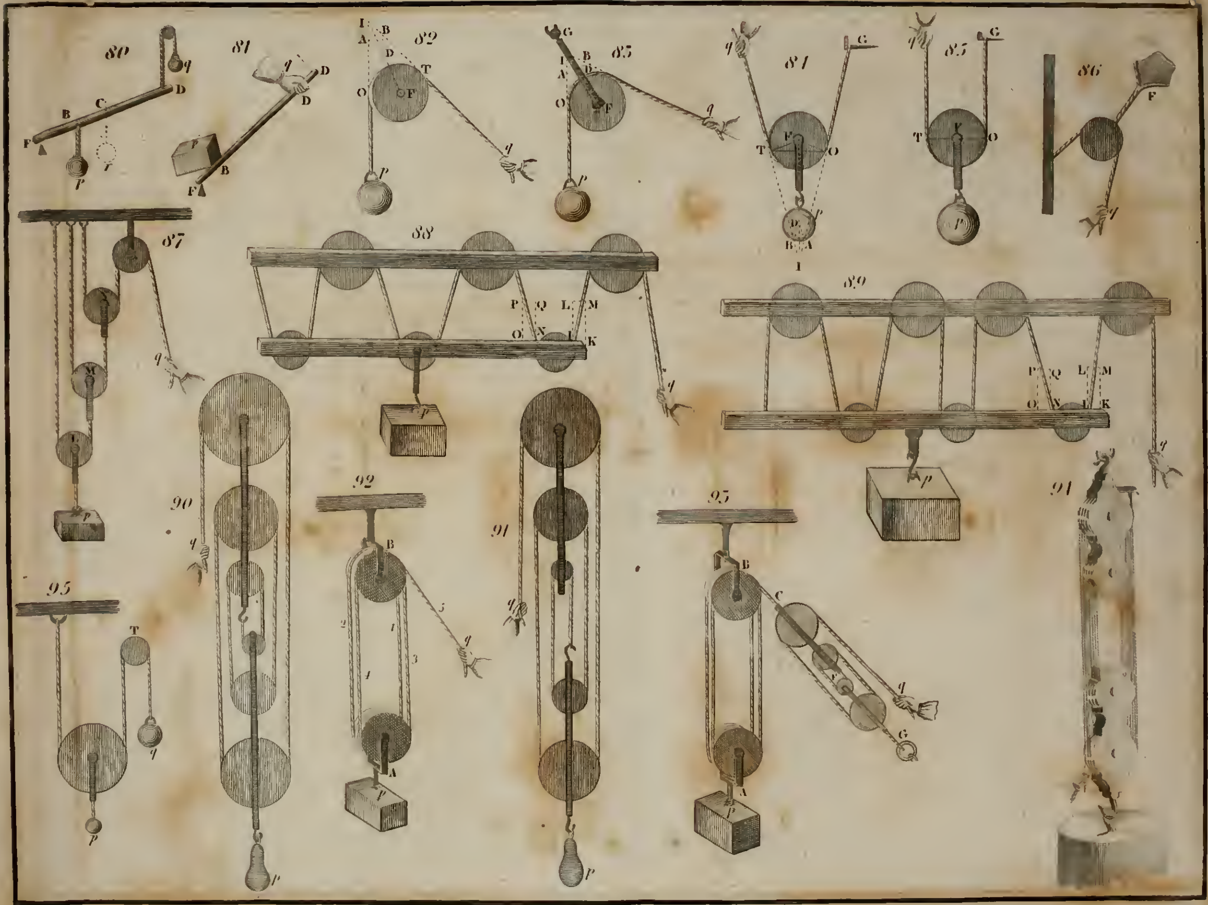

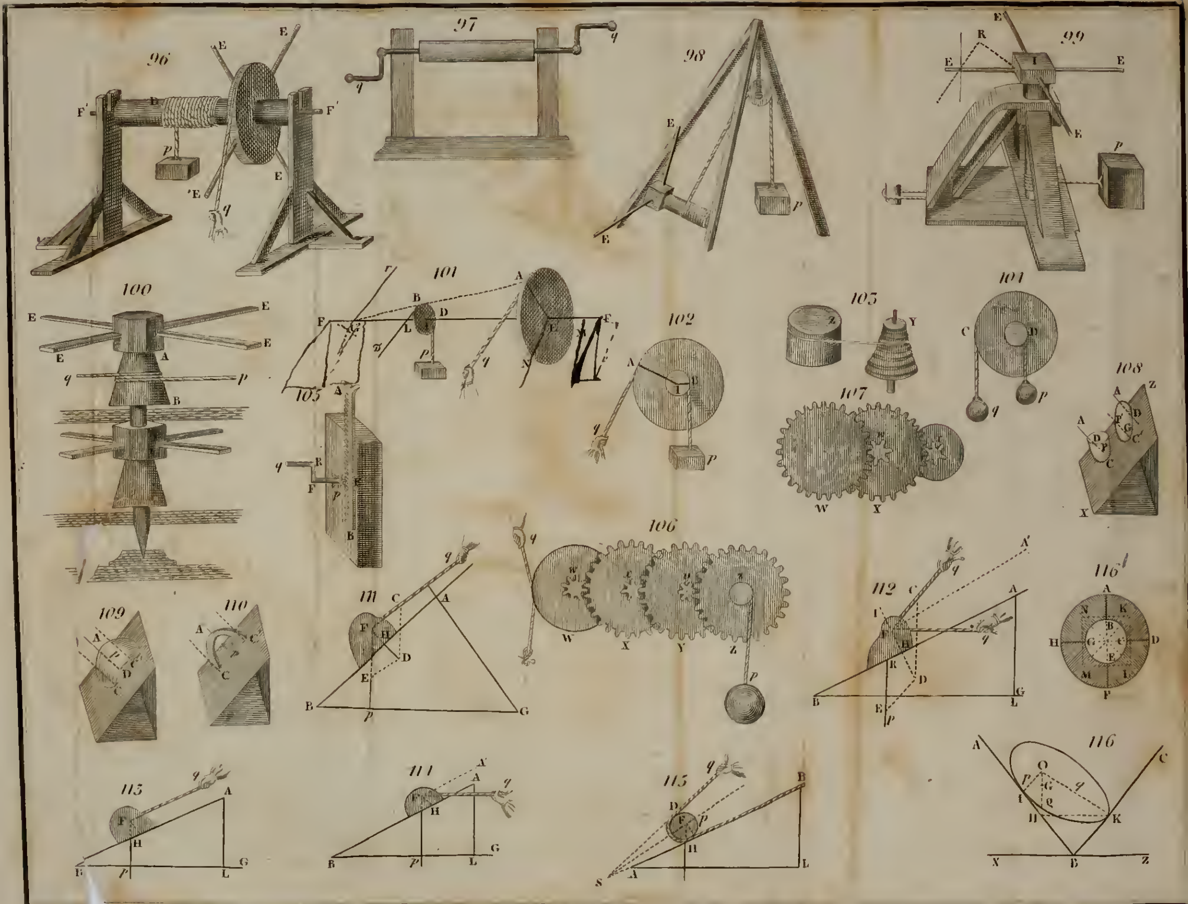
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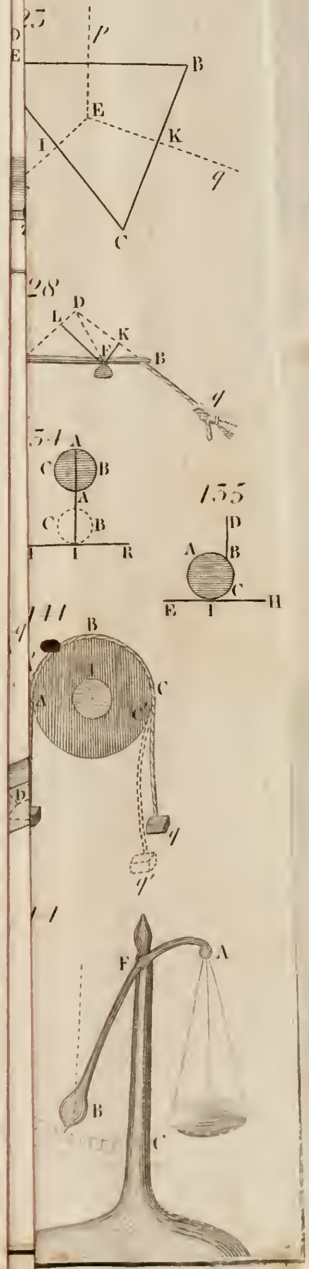

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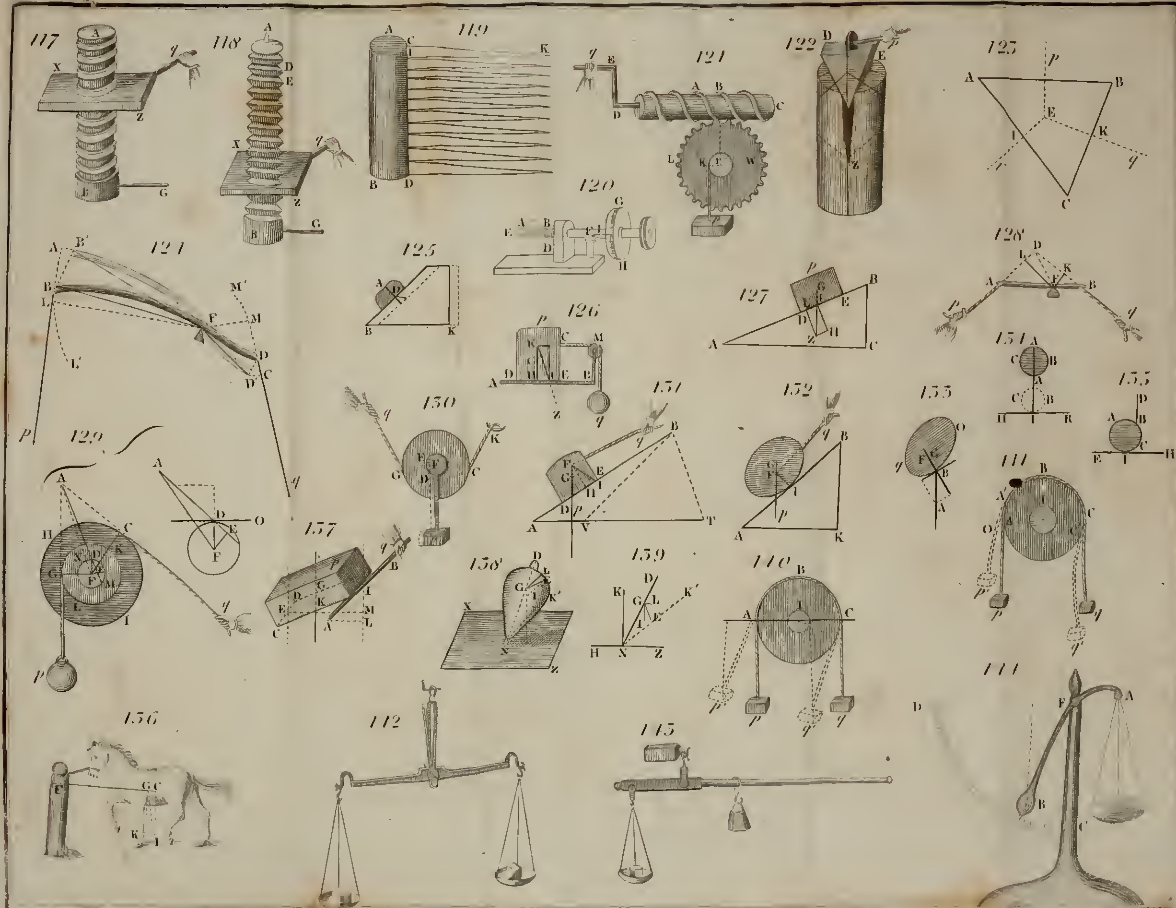
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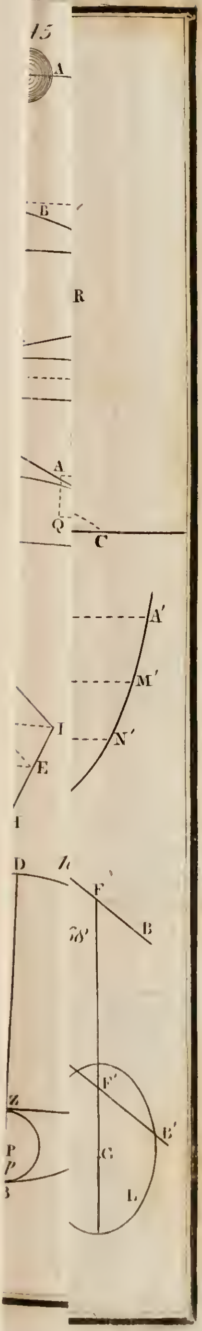

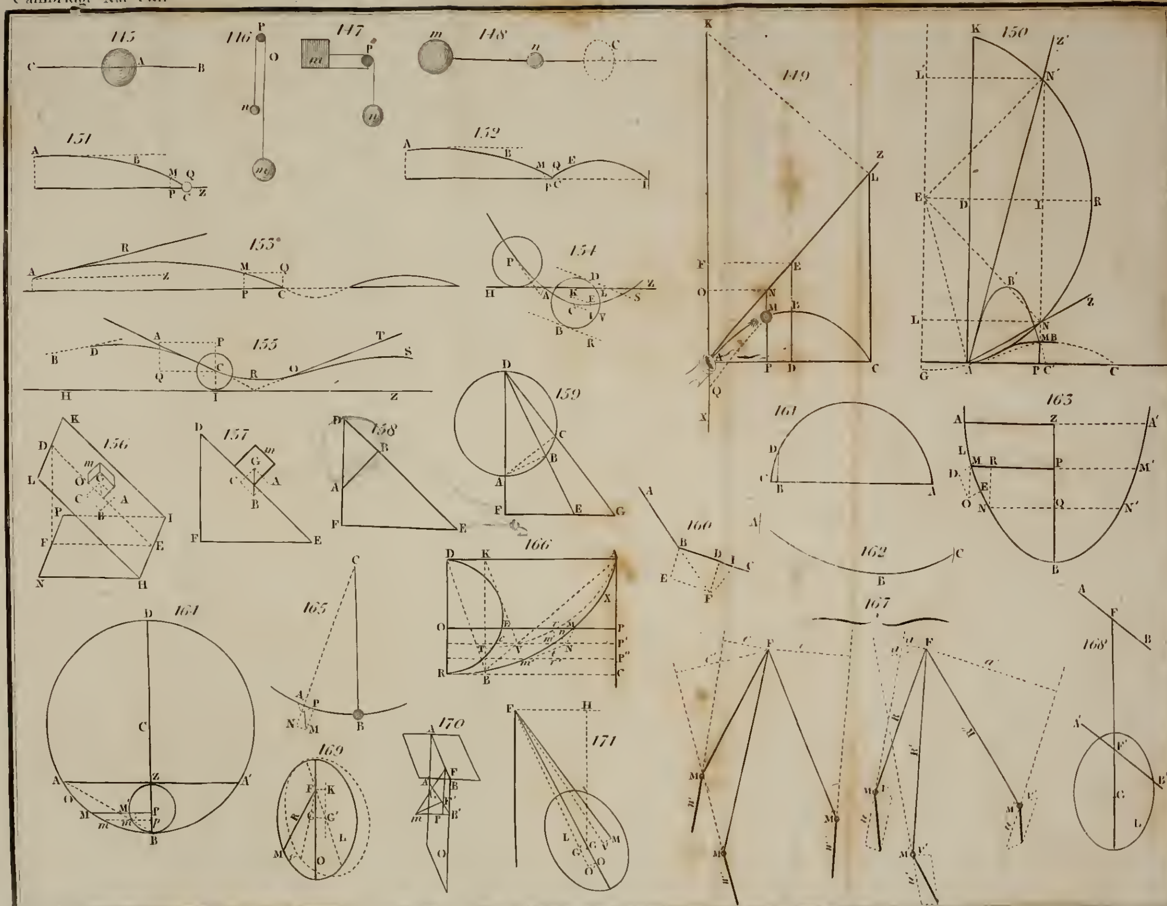

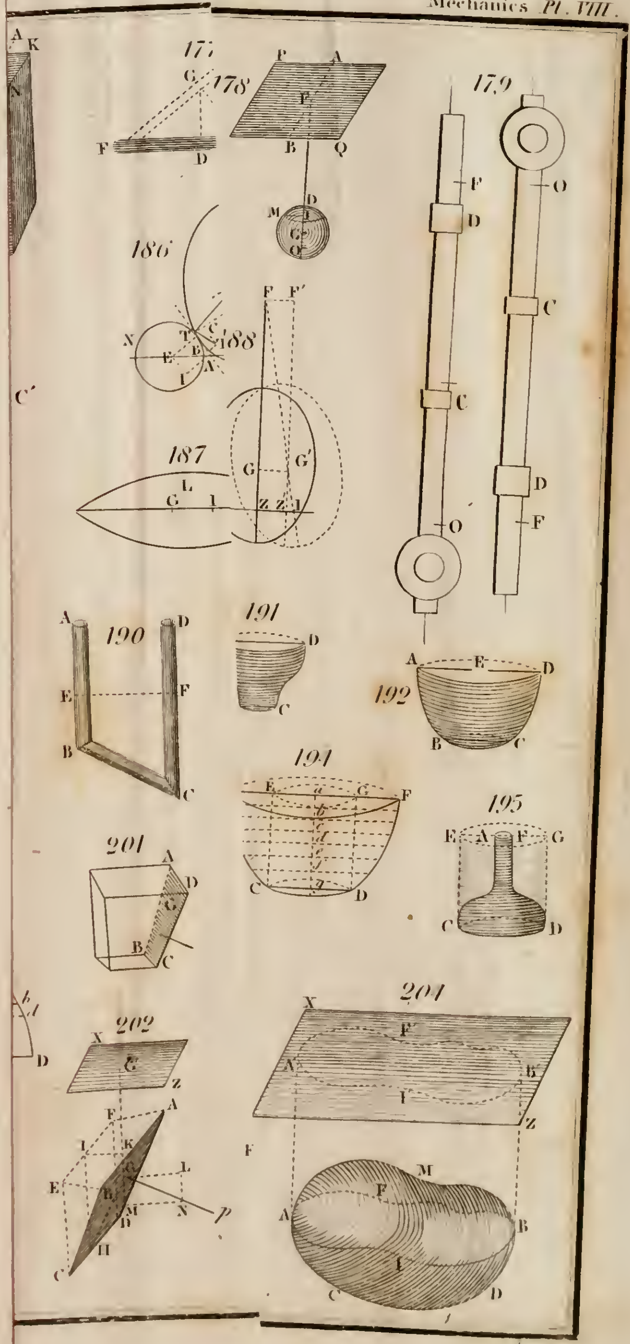
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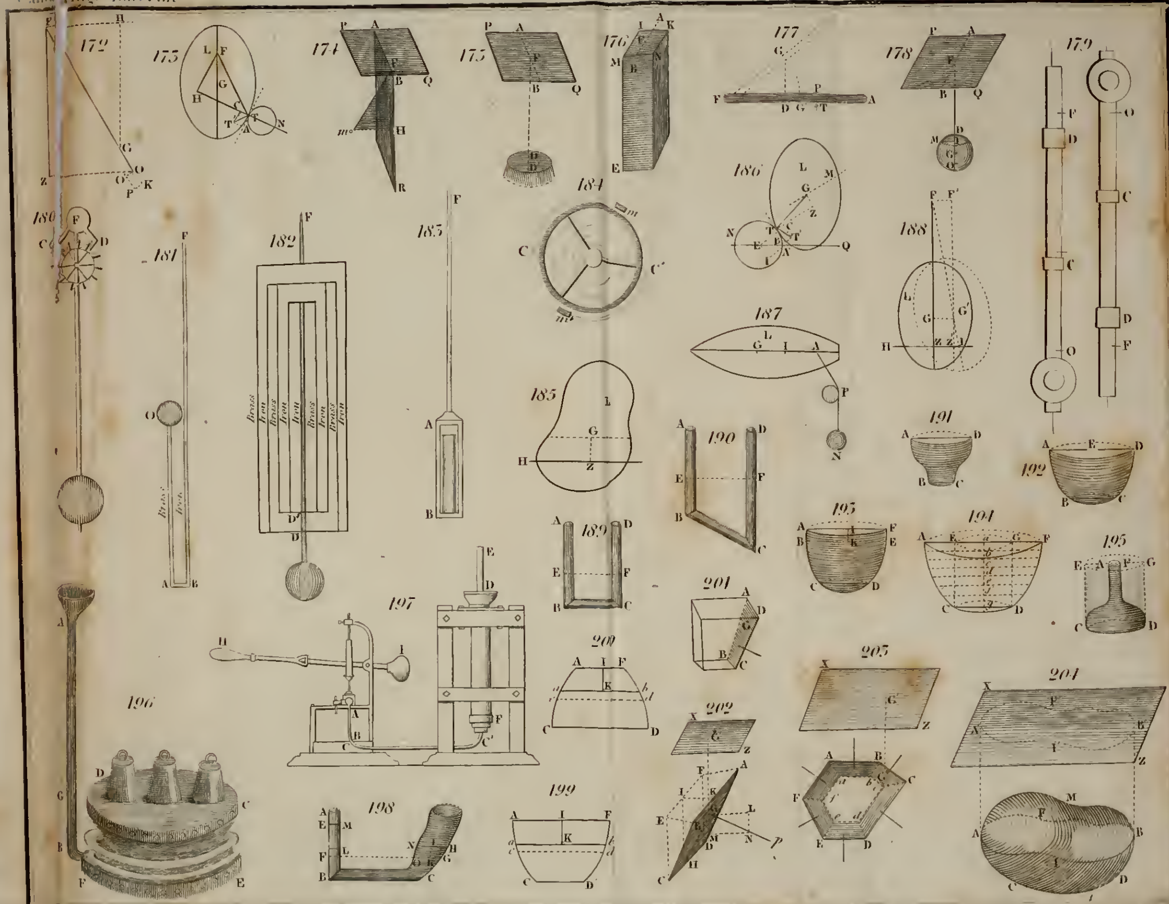
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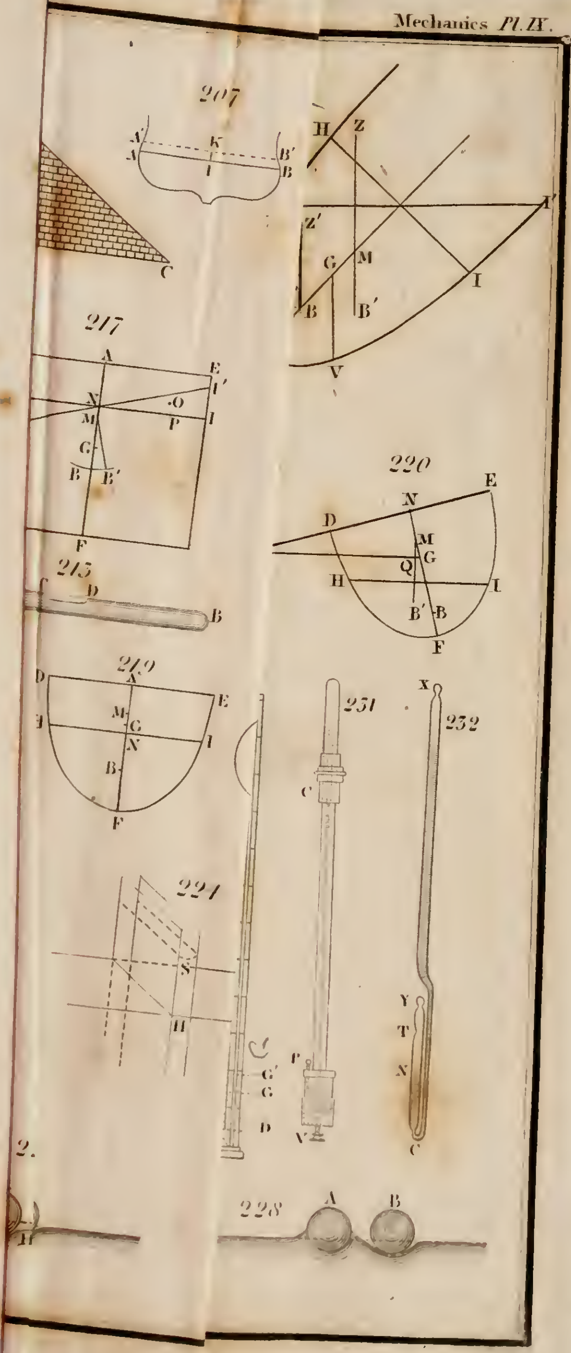
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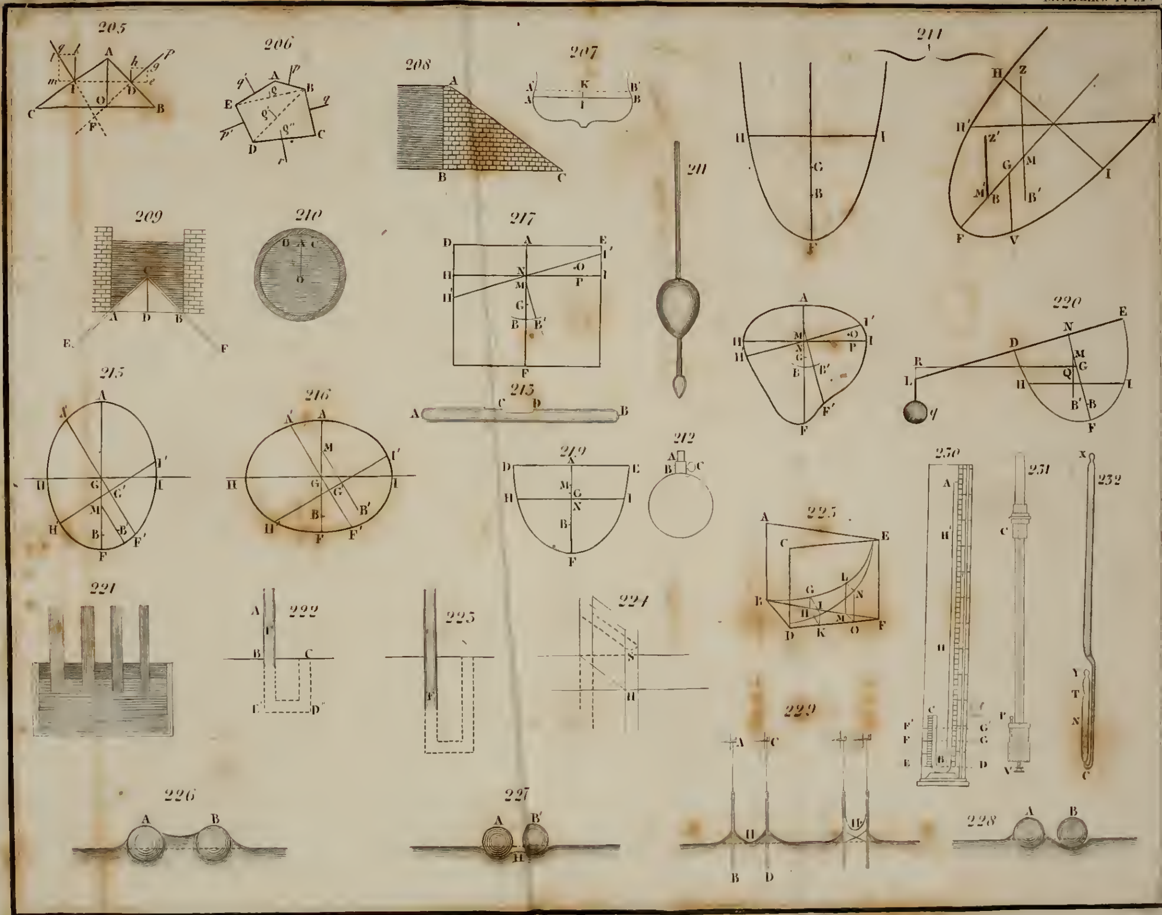
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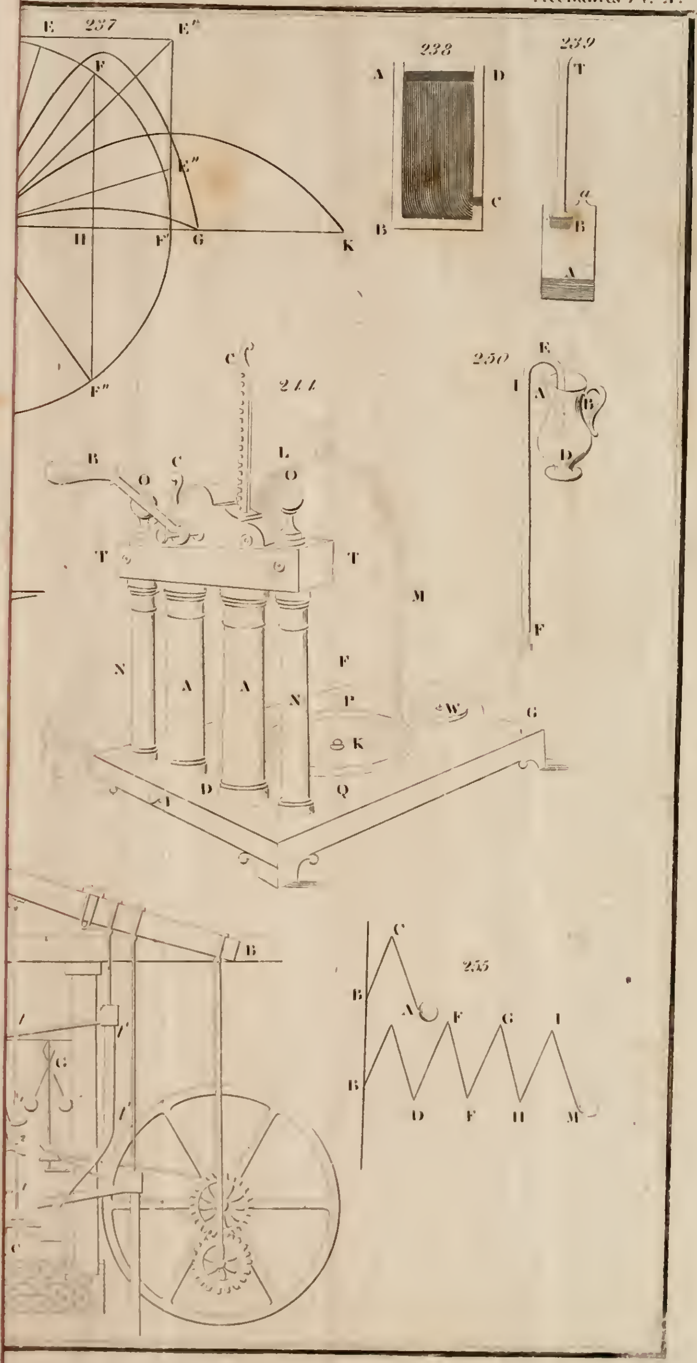
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